Algebra Qualifying Exam
January 2012

Do all 5 problems.

1. Let G be a finite group of order $4312 = 2^3 \cdot 7^2 \cdot 11$.
 (a) Show that G has a subgroup of order 77. (4 points)
 (b) Prove that G has a subgroup of order 7 whose normalizer in G has index dividing 8. (4 points)
 (c) Conclude that G is not simple. (2 points)

2. Let R be a commutative ring with 1, and let Q be a primary ideal of R. Suppose that

$$Q = \bigcap_{i=1}^{k} X_i$$

is a finite intersection of the ideals X_i.
 (a) If each X_i is a prime ideal, prove that $Q = X_j$ for some j. (Hint. You might wish to first prove that Q is prime.) (5 points)
 (b) Now suppose that R is Noetherian and that each X_i is primary. Assume also that the radicals of the X_i are distinct. Again show that $Q = X_j$ for some j. (5 points)

3. Let $K \subseteq F \subseteq E$ be fields with $|E : F| < \infty$, and let A be the subfield of E consisting of all elements of E that are algebraic over K. Assume that $F \cap A = K$.
 (a) If $\alpha \in A$ and $f(x)$ is the monic minimal polynomial of α over F, show that all coefficients of $f(x)$ lie in K. (4 points)
 (b) Now assume that K has characteristic 0. If B is an intermediate field with $K \subseteq B \subseteq A$ and $|B : K| < \infty$, prove that $|B : K| \leq |E : F|$. (4 points)
 (c) Conclude that $|A : K| \leq |E : F|$. (2 points)

4. Let V be a nonzero finite-dimensional vector space over the complex numbers.
 (a) If S and T are commuting linear operators on V, prove that each eigenspace of S is mapped into itself by T. (2 points)
 (b) Now let $A_1, A_2, \ldots, A_k$ be finitely many linear operators on V that commute pairwise. Prove that they have a common eigenvector in V. (4 points)
 (c) If V has dimension n, show that there exists a nested sequence of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

where each V_j has dimension j and is mapped into itself by each of the operators $A_1, A_2, \ldots, A_k$. (4 points)

5. Let K be a field and assume that -1 is not a square in K. Let $G = \mathrm{GL}(2, K)$ be the group of invertible 2×2 matrices over K.
 (a) If $g \in G$, show that g has order 4 if and only if $\det(g) = 1$ and $\mathrm{tr}(g) = 0$. (5 points)
 (b) Find explicitly an element $g \in G$ of order 4. (1 point)
 (c) Suppose there exist elements $a, b \in K$ with $a^2 + b^2 = -1$. Show that G contains two elements g, h of order 4 such that the product gh also has order 4. (4 points)