Algebra Qualifying Exam
January 1991

Do all 5 problems.

1. Let G be a finite group having exactly n Sylow p-subgroups for some prime p. Show that there exists a subgroup H of the symmetric group Sym_n such that H also has exactly n Sylow p-subgroups.

2. Let R be a commutative integral domain and let $R[x_1, x_2, \ldots, x_n]$ be the polynomial ring over R in the n variables $x_1, x_2, \ldots, x_n$. If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is an n-tuple of elements of R, then there is an evaluation homomorphism $\varphi_{\mathbf{a}}\colon R[x_1, x_2, \ldots, x_n] \to R$ given by $\varphi_{\mathbf{a}}(f) = f(a_1, a_2, \ldots, a_n)$.

a. (5 points) If R is the field of complex numbers and if I is a proper ideal of $R[x_1, x_2, \ldots, x_n]$, show that there exists an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ with $\varphi_{\mathbf{a}}(I) \neq R$.

b. (5 points) Now let R be the ring of integers and let I be the ideal of the polynomial ring $R[x]$ in one variable generated by 3 and $x^2 + 1$. Show that I is a proper ideal of $R[x]$ but that $\varphi_{\mathbf{a}}(I) = R$ for all 1-tuples $\mathbf{a} = (a)$ with $a \in R$.

3. Let $F \subseteq E$ be an extension of fields of characteristic $\neq 2$ and assume that the degree $|E : F| = 4$.

a. (3 points) Show that $E = F[\alpha]$ for some α.

b. (2 points) If $E = F[\alpha]$, where α is a root of a polynomial of the form $x^4 + ax^2 + b \in F[x]$, prove that there exists an intermediate field properly between E and F.

c. (5 points) Now let $E = F[\beta]$ with no assumption on β and let $L \supseteq E$ be a splitting field for the minimal polynomial of β over F. If $\mathrm{Gal}(L/F)$ is isomorphic to the symmetric group Sym_4, show that there is no intermediate field properly between E and F.

4. Let F be an algebraically closed field of prime characteristic p and let V be an F-vector space of dimension precisely p. Suppose A and B are linear operators on V such that $AB - BA = B$. If B is nonsingular, prove that V has a basis $\{ v_1, v_2, \ldots, v_p \}$ of eigenvectors of A such that $Bv_i = v_{i+1}$ for $1 \leq i \leq p - 1$ and $Bv_p = \lambda v_1$ for some $0 \neq \lambda \in F$.

5. Let $G \neq \langle 1 \rangle$ be a possibly infinite group whose subgroups are linearly ordered by inclusion. In other words, if H and K are subgroups of G, then either $H \subseteq K$ or $K \subseteq H$.

a. (5 points) Prove that G is an abelian group and that the orders of the elements of G are all powers of the same prime p.

b. (5 points) If $G_n = \{ g \in G \mid g^{p^n} = 1 \}$, prove that $|G_n| \leq p^n$.