Algebra Qualifying Exam
January 1992

Do all 5 problems.

1. Let G be a finite group and fix a prime number p. Define the function f on the set of subgroups $H \subseteq G$ by

$$f(H) = |\{\, P \in \mathrm{Syl}_p(G) \mid P \supseteq H \,\}|.$$

In other words, $f(H)$ is the number of Sylow p-subgroups of G which contain H. Prove that if $f(H) > 0$, then $f(H) \equiv 1 \bmod p$.

2. Let F be a field and let R be the ring of all 3×3 matrices over F with $(3,1)$ and $(3,2)$ entry equal to 0. Thus,

$$R = \begin{pmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{pmatrix}.$$

Prove that the Jacobson radical of R is a minimal left ideal, but not a minimal right ideal.

3. Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose E is a splitting field for $f(x)$ over F and assume that there exists an element $\alpha \in E$ such that both α and $\alpha + 1$ are roots of $f(x)$.

i. Show that the characteristic of F is not zero. (5 points)

ii. Prove that there exists a field L between F and E such that the degree $|E : L|$ is equal to the characteristic of F. (5 points)

4. Let V be a finite dimensional complex vector space and suppose $\langle\ ,\ \rangle \colon V \times V \to \mathbb{C}$ is an inner product on V, that is, $\langle\ ,\ \rangle$ is a positive definite Hermitian form on V.

i. Suppose $T \colon V \to V$ is a linear transformation such that $\langle Tv, v\rangle = 0$ for all $v \in V$. Prove that $V = 0$. (7 points)

ii. Does the result of part (i) hold if V is assumed to be a real inner product space? Justify your answer. (3 points)

5. Let $\mathbb{Z}$ denote the ring of integers and let $\mathbb{Q}$ and $\mathbb{C}$ be the rational and complex fields, respectively. If $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$, then we let $\mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_n]$ denote the ring generated by these elements over $\mathbb{Z}$. In particular, note that $\mathbb{Z}[1/2]$ is the set of all rational numbers with denominator a power of 2. Now suppose that $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of the integer polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in \mathbb{Z}[x]$ with $a_0 = 2$.

i. Prove that $2\alpha_i$ is an algebraic integer for all $i = 1, 2, \dots, n$. (3 points)

ii. Show that $\mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_n] \cap \mathbb{Q} \subseteq \mathbb{Z}[1/2]$. (4 points)

iii. If some a_j with $j \geq 1$ is odd, prove that $1/2 \in \mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_n] \cap \mathbb{Q}$ and deduce that the latter intersection is equal to $\mathbb{Z}[1/2]$. What happens if all a_j are even? (3 points)