Algebra Qualifying Exam
January 1993

Do all 5 problems.

1. Let $G = A \dot{\times} B$ be the internal direct product of finite subgroups A and B. Suppose H is a subgroup of G with $A \cap H = 1$.

i. Show that B contains a subgroup isomorphic to H, but that B need not contain H in general. (5 points)

ii. If A and B have relatively prime orders, prove that $H \subseteq B$. (5 points)

2. Let $K \subseteq E$ be an extension of fields and let R be the subring of the polynomial ring $E[x]$ consisting of all polynomials with constant term in K. In other words,

$$R = K + Ex + Ex^2 + Ex^3 + \cdots$$

Now let I be a nonzero ideal of R and let m be the minimal degree of the nonzero elements of I. Define

$$I_m = \{\, f(x) \in I \mid \deg f(x) = m \,\} \cup \{\, 0 \,\}$$

so that I_m is clearly a nonzero K-subspace of R.

i. If $\dim_K I_m = 1$, prove that I is a principal ideal. (4 points)

ii. If $\dim_K I_m > 1$, show that I_m contains a nonzero polynomial with constant term equal to 0. (2 points)

iii. If I is a prime ideal which is not principal, prove that $m = 1$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $K = \mathbb{Q}[\sqrt{2}\,]$. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic irreducible polynomial of *odd* degree $n \geq 1$ and notice that $f(x + \sqrt{2}\,)$ is a monic polynomial of degree n in $K[x]$.

i. Show that the coefficient of x^{n-1} in $f(x + \sqrt{2}\,)$ is not rational. (2 points)

ii. Show that the polynomial $f(x + \sqrt{2}\,)$ is irreducible in $K[x]$. (4 points)

iii. Prove that the polynomial $g(x) = f(x + \sqrt{2}\,)f(x - \sqrt{2}\,)$ is irreducible in the ring $\mathbb{Q}[x]$. (4 points)

4. Let $V \neq 0$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let X, Y, Z be linear operators on V which satisfy

$$XY - YX = Z \qquad XZ = ZX \qquad YZ = ZY.$$

If V has no proper subspace invariant under all three operators, prove that $\dim_{\mathbb{C}} V = 1$.

5. Let V be a vector space over the rational numbers $\mathbb{Q}$ and let $v_1, v_2, \ldots, v_n \in V$. Show that there exist elements $w_1, w_2, \ldots, w_m \in V$ which are linearly independent over $\mathbb{Q}$ and which satisfy

$$\sum_{i=1}^{n} v_i \mathbb{Z} = \sum_{j=1}^{m} w_j \mathbb{Z}$$

where $\mathbb{Z}$ is the ring of integers.