Algebra Qualifying Exam
January 1994

Do all 5 problems.

1. A finite group is said to be *perfect* if it has no nontrivial abelian homomorphic image.

i. Show that a perfect group has no nontrivial solvable homomorphic image. (3 points)

ii. Let $H \triangleleft G$ with G/H perfect. If $\theta\colon G \to S$ is a homomorphism from G to a solvable group S and if $N = \ker\theta$, prove that $G = NH$ and deduce that $\theta(H) = \theta(G)$. (7 points)

2. Let R be a ring and let V be a right R-module. Assume that every simple submodule of V is a direct summand of V.

i. If W is any submodule of V, show that any simple submodule of W is a direct summand of W. (5 points)

ii. If V is an Artinian module, that is if its submodules satisfy the minimal condition, prove that V is a direct sum of finitely many simple submodules. (5 points)

3. Let α be the real positive 16th root of 3 and consider the field $F = Q[\alpha]$ generated by α over the rationals Q. Notice that we have the chain of intermediate fields

$$Q \subseteq Q[\alpha^8] \subseteq Q[\alpha^4] \subseteq Q[\alpha^2] \subseteq Q[\alpha] = F.$$

i. Compute the degrees of these five intermediate fields over Q and conclude that these fields are all distinct. (4 points)

ii. Show that every intermediate field between Q and F is one of the above. Hint. If $Q \subseteq K \subseteq F$, consider the constant term of the minimal polynomial of α over K. (6 points)

4. Let X be a subspace of $\mathrm{M}_n(C)$, the C-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in X is invertible. Prove that $\dim_C X \leq 1$.

5. Let E be an algebraic extension of the rational numbers Q and let $\alpha \in E$.

i. Prove that there exists a nonzero integer $n \in Z$ such that $n\alpha$ is an algebraic integer. (4 points)

ii. Show that $Z[\alpha]$ does not contain Q and hence conclude that $Z[\alpha]$ is not a field. (6 points)