

Algebra Qualifying Exam
January 1995

Do all 5 problems.

1. Let P be a Sylow p -subgroup of a finite group G and let $N = \mathbb{N}_G(P)$ be its normalizer.
 - i. Show that N is not contained in any proper normal subgroup of G . (5 points)
 - ii. If the commutator subgroup G' is abelian, prove that $G' \cap N$ is normal in G . (5 points)

2. Let R be a commutative Noetherian domain and suppose that P is the unique nonzero prime ideal of R .
 - i. Show that every element of R not in P is a unit of R . (3 points)
 - ii. If Q is a nonzero ideal of R , prove that Q is primary and that $Q \supseteq P^n$ for some integer $n \geq 1$. (3 points)
 - iii. If $P = (\pi)$ is principal, prove that every nonzero element of R is a product of a unit and a power of π . (4 points)

3. Let \mathbb{Q} be the field of rational numbers and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathbb{Q}[x]$. Suppose E is a splitting field for $f(x)$ over \mathbb{Q} and set $G = \text{Gal}(E/\mathbb{Q})$.
 - i. Find $|E : \mathbb{Q}|$ and determine the Galois group G up to isomorphism. (5 points)
 - ii. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of G on Ω . (5 points)

4. Let A be an $n \times n$ matrix over an algebraically closed field K and let $K[A]$ denote the K -linear span of the matrices $I = A^0, A, A^2, A^3, \dots$. Show that A is diagonalizable if and only if $K[A]$ contains no nonzero nilpotent element.

5. Let G be a (not necessarily finite) group and denote the operation in G by multiplication. Let $\mathbb{Z}[G]$ denote the group ring of G over the integers \mathbb{Z} . Thus, every element of $\mathbb{Z}[G]$ is a finite \mathbb{Z} -linear combination of elements of G , and the multiplication in $\mathbb{Z}[G]$ is built naturally from the multiplication in G . Let I be a right ideal of $\mathbb{Z}[G]$ and define
$$\text{gp}(I) = \{g \in G \mid 1 - g \in I\}.$$
 - i. Prove that $\text{gp}(I)$ is a subgroup of G . (4 points)
 - ii. If I is a 2-sided ideal of $\mathbb{Z}[G]$, show that $\text{gp}(I)$ is normal in G . (3 points)
 - iii. If $\text{gp}(I) = G$, prove that I is a 2-sided ideal of $\mathbb{Z}[G]$. (3 points)