Algebra Qualifying Exam
January 1996

Do all 5 problems.

1. A finite group G is said to have property $*$ if there exists a conjugacy class $\mathcal{K}$ of G such G is generated by the elements of $\mathcal{K}$.
   i. If G has property $*$, prove that G/G' is cyclic, where G' is the commutator subgroup of G. (3 points)
   ii. Show that G has property $*$ if and only if G is not the set-theoretic union of its proper normal subgroups. (3 points)
   iii. Suppose $G = N \times M$ where N is nonabelian simple and M has property $*$. Prove that G has property $*$. (Hint. First show that every normal subgroup of G that does not contain N must be contained in M.) (4 points)

2. Let R be a commutative ring with 1 and suppose that M is an ideal of R.
   i. If M is both maximal and principal, show that there is no ideal I of R satisfying $M > I > M^2$, where $>$ denotes strict inclusion. (6 points)
   ii. Give examples to show that neither of the two conditions on M in part (i) can be removed. (4 points)

3. Let $Q \subseteq L \subseteq E$ be fields with Q the rational numbers and with $|E : L| < \infty$. Let K be the subfield of E consisting of all those elements of E which are algebraic over Q, and assume that $K \cap L = Q$.
   i. If $\alpha \in K$, show that its minimal polynomial over Q is irreducible over L and deduce that $|Q[\alpha] : Q| \leq |E : L|$. (5 points)
   ii. Show that $|K : Q| \leq |E : L|$. (Hint. Start with a subfield M of K maximal with the property that $|M : Q| \leq |E : L|$.) (5 points)

4. Let V be a vector space over a field K and let S and T be K-linear operators on V. Suppose that S is one-to-one, that $T(v) = 0$ for some $0 \neq v \in V$, and that $TS - ST = S$.
   i. For every integer $n \geq 0$, prove that $S^n(v)$ is an eigenvector for T and determine the corresponding eigenvalue. (4 points)
   ii. If K has characteristic 0, prove that $\dim_K V = \infty$. (4 points)
   iii. If K has characteristic $p > 0$, then $\dim_K V$ can be finite. Give a concrete example of such a finite-dimensional situation when $p = 3$. (2 points)

5. Let q be a prime power and let F be the finite field of size q. Let

$$f(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \in F[x].$$

   i. If $f(x)$ has a root in F, show that $f(x)$ splits completely over F, and prove that this happens precisely when $q \equiv 0$ or $1 \bmod 5$. (6 points)
   ii. If $f(x)$ has an irreducible monic factor $g(x)$ of degree 2, show that $g(x)$ has constant term equal to 1. (2 points)
   iii. Factor $f(x)$ explicitly into irreducible quadratic factors when $q = 29$. (2 points)