Algebra Qualifying Exam
January 1998

Do all 5 problems.

1. Fix a prime p and let G be a finite group with the property that every nonidentity p-subgroup of G is contained in a unique Sylow p-subgroup of G. Suppose $N \triangleleft G$ and $|N|$ is divisible by p.
 i. If P and Q are Sylow p-subgroups of G, show that $Q = P^n$ for some element $n \in N$. (6 points)
 ii. Prove that G/N has a unique Sylow p-subgroup. (4 points)

2. Let R be a commutative domain and write (a) for the principal ideal generated by $a \in R$. Recall that an element of R is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.
 i. Show that $(a) \subseteq (b)$ if and only if $b \mid a$, and that $(a) = (b)$ if and only if $b = au$ for some unit $u \in R$. (2 points)
 ii. If R is a UFD (unique factorization domain), prove that the set of principal ideals of R satisfies the maximal condition. (4 points)
 iii. If the set of principal ideals of R satisfies the maximal condition, show that every nonzero, nonunit element of R can be written as a finite product of irreducible elements. (4 points)

3. Let p be a prime, let $F \subseteq K$ be fields of characteristic 0, and assume that F contains a primitive pth root of unity. Fix $a \in K$.
 i. Prove that there exists a field $E \supseteq K$ such that E contains a pth root of a and $|E : K| = 1$ or p. (4 points)
 ii. Now assume that K is a finite degree Galois extension of F. Show that there exists a field $E \supseteq K$ such that E contains a pth root of a, E is Galois over F, and $|E : K|$ is a power of p. (6 points)

4. Let V be a finite dimensional vector space over a field of characteristic 0. Suppose $T: V \to V$ is a linear operator such that the trace $\operatorname{tr} T^k = 0$ for all integers $k \geq 1$.
 i. Show that the constant term of the characteristic polynomial of T is zero, and deduce that $T(V) \neq V$. (5 points)
 ii. Let S denote the restriction of T to the subspace $T(V)$, so that S is a linear operator on $T(V)$. Prove that $\operatorname{tr} S^k = 0$ for all integers $k \geq 1$. (4 points)
 iii. Show that T is nilpotent. (1 point)

5. Let G be a (not necessarily finite) group and let $\theta: G \to G$ be a homomorphism such that $\theta^n(G) = \{ 1 \}$ for some integer $n \geq 1$.
 i. If the kernel of θ is finite, prove that the kernel of θ^2 is finite, and deduce that G is finite. (5 points)
 ii. If $\theta(G)$ has finite index in G, prove that $\theta^2(G)$ has finite index in G, and deduce that G is finite. (5 points)