Algebra Qualifying Exam
January 1999

Do all 5 problems.

1. Let G be a finite group and let $A \subseteq G$ be a maximal (proper) subgroup. Assume that A is abelian, that $|G : A| = p^n$ for some prime p, and that A contains no nonidentity normal subgroup of G.
   i. Show that p does not divide $|A|$. (4 points)
   ii. Prove that the set S of elements of G not conjugate to any nonidentity element of A has cardinality precisely p^n. (5 points)
   iii. Show that G is not simple. (1 point)

2. Let R be a ring with 1, and recall that R is naturally a right R-module with respect to right multiplication. We denote this right regular R-module by R_R.
   i. Prove that R is a division ring if and only if R_R is a simple R-module. (5 points)
   ii. Prove that R is a division ring if and only if every nonzero right R-module contains a submodule isomorphic to R_R. (5 points)

3. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial over the rational numbers $\mathbb{Q}$, and let α and β be roots of $f(x)$ in the complex numbers $\mathbb{C}$. Suppose $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$ where E is a (finite) Galois extension of $\mathbb{Q}$.
   i. Show that $\mathbb{Q}[\alpha] \cap E$ is isomorphic to $\mathbb{Q}[\beta] \cap E$. (5 points)
   ii. Now assume that $E = \mathbb{Q}[\varepsilon]$ where ε is a root of unity. Prove that $\mathbb{Q}[\alpha] \cap E = \mathbb{Q}[\beta] \cap E$. (5 points)

4. Let V be a finite-dimensional vector space over an algebraically closed field F, and let S and T be commuting linear operators on V. Assume that the characteristic polynomial of S has distinct roots.
   i. Show that every eigenvector of S is an eigenvector for T. (5 points)
   ii. If T is nilpotent, prove that $T = 0$. (5 points)

5. Let R be a ring with 1 and let V be a right R-module. Assume that $M_1, M_2, \ldots, M_n$ are finitely many R-submodules of V with $M_1 \cap M_2 \cap \cdots \cap M_n = 0$, and let W be the (external) direct sum

$$W = V/M_1 \oplus V/M_2 \oplus \cdots \oplus V/M_n.$$

   i. Show that V is isomorphic to an R-submodule of W. (4 points)
   ii. Now suppose in addition that the modules V/M_i are simple and pairwise nonisomorphic. Prove that V is isomorphic to W. (Hint. First observe that W has a composition series of length n.) (6 points)