INVARIANT IDEALS OF ABELIAN GROUP ALGEBRAS UNDER THE TORUS ACTION OF A FIELD, I

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Happy Birthday, Susan

Abstract. Let $V = V_1 \oplus V_2$ be a finite-dimensional vector space over an infinite locally-finite field F. Then V admits the torus action of $G = F^\bullet$ by defining $(v_1 \oplus v_2)^g = v_1 g^{-1} \oplus v_2 g$. If K is a field of characteristic different from that of F, then G acts on the group algebra $K[V]$ and it is an interesting problem to determine all G-stable ideals of this algebra. In this paper, we consider the special case when V_1 and V_2 are both 1-dimensional and we show that there are just four G-stable proper ideals of $K[V]$.

1. Introduction

Let K be a field, let V be an abelian group, and let G be a group of automorphisms of V. Then G acts on the group algebra $K[V]$ and it is an interesting, and surprisingly difficult, problem to describe the G-stable ideals of $K[V]$. The motivation for this actually comes from the study of the lattice of ideals in group algebras of certain infinite locally finite groups. This can be seen, for example, in the survey [6] or in the introduction to paper [7], but we will not expand upon this theme here. A natural special case of the problem occurs when V is a vector space over an infinite field F and when $G = F^\bullet$ acts on V by scalar multiplication. This turns out to have a rather beautiful solution [1, 7, 4], especially when V is finite dimensional. Indeed, one can even allow F to be a division algebra.

The next case of interest surely arises by introducing inverses from F. Specifically, let V_1 and V_2 be two vector spaces over F, form $V = V_1 \oplus V_2$, and let $G = F^\bullet$ act on V by $(v_1 \oplus v_2)^g = v_1 g^{-1} \oplus v_2 g$. We call this the *torus action* of F by analogy to the way the torus in $\mathrm{SL}_2(F)$ acts on F^2. The goal of this paper is to describe the G-stable ideals of $K[V]$ when $\dim_F V_1 = \dim_F V_2 = 1$ and when F is an infinite locally finite field. As we will see, there are just four G-stable proper ideals in this situation. The argument here is tricky, but not difficult, and is hopefully a first step towards a solution of the general problem.

Let V be an abelian group, viewed multiplicatively, and let $K[V]$ denote its group algebra over the field K. If A is a subgroup of V, then there exists a natural epimorphism $K[V] \to K[V/A]$ and we let $\omega(A; V) = \omega_K(A; V)$, the *augmentation ideal* of A in V, denote its kernel. Thus, $\omega(A; V)$ is the K-linear span of all elements of the form $(1 - a)v$ with $a \in A$ and $v \in V$, and clearly

$$A = \{v \in V \mid 1 - v \in \omega(A; V)\}.$$

2000 *Mathematics Subject Classification.* 16S34, 12E20.

Research supported in part by an NSA grant.

Observe that if A and B are subgroups of V and if $C = \langle A, B\rangle$ is the group they generate, then $\omega(A;V) + \omega(B;V) = \omega(C;V)$. Indeed, if I denotes the ideal $\omega(A;V) + \omega(B;V)$, then surely $I \subseteq \omega(C;V)$. On the other hand, both A and B are contained in the kernel of the homomorphism $K[V] \to K[V]/I$ restricted to the group V, and hence C is also contained in this kernel. Now, if G is a group that acts as automorphisms on V, then G also acts on $K[V]$, and it is clear that A is a G-stable subgroup of V if and only if $\omega(A;V)$ is a G-stable ideal of $K[V]$.

We now return to additive notation for V. The main result of this paper is

Theorem 1.1. *Let F be an infinite locally-finite field, let V_1 and V_2 be 1-dimensional F-vector spaces, and set $V = V_1 \oplus V_2$. If $G = F^\bullet$, then we can let G act as the torus on V, and hence G acts on the group algebra $K[V]$. Suppose, in addition, that $\operatorname{char} K \neq \operatorname{char} F$. Then the nontrivial G-stable ideals of $K[V]$ are precisely $\omega(V;V)$, $\omega(V_1;V)$, $\omega(V_2;V)$ and $\omega(V_1;V) \cap \omega(V_2;V)$.*

Note that, if V is a torsion abelian group having no elements of order equal to the characteristic of K, then $K[V]$ is a commutative von Neumann regular algebra (see [5, Theorem 1.1.5]). It follows that if $I, J \triangleleft K[V]$, then $I \cap J = IJ$. In particular, finite products and finite intersections of ideals coincide here. Furthermore, every ideal of $K[V]$ is semiprime.

We close this section by describing the G-stable subgroups of $V = V_1 \oplus V_2$ when V_1 and V_2 are arbitrary F-vector spaces. The argument is slightly simpler in the case of locally-finite fields, but we prove the result in full generality.

Lemma 1.2. *Let F be a field with $|F| \geq 5$. If $V = V_1 \oplus V_2$ is an F-vector space admitting the torus action of $G = F^\bullet$, then the G-stable subgroups of V are precisely those of the form $A \oplus B$, where A is an F-subspace of V_1 and B is an F-subspace of V_2.*

Proof. Obviously, all such $A \oplus B$ are G-stable subgroups of V. We consider the converse. To this end, Let $W_1 = W_2 = F$ be the 1-dimensional G-modules given by $w_1^g = w_1 g^{-1}$ and $w_2^g = w_2 g$ for all $w_1 \in W_1$, $w_2 \in W_2$ and $g \in G = F^\bullet$. Then W_1 and W_2 are both irreducible G-modules since any G-submodule of W_i is closed under addition and scalar multiplication by F. We claim now that W_1 and W_2 are not G-isomorphic. Indeed, if such an isomorphism $\theta\colon W_1 \to W_2$ exists, then for all nonzero elements $f \in F$, we have $\theta(f) = \theta(1\cdot f) = \theta(1^{f^{-1}}) = \theta(1)^{f^{-1}} = \theta(1)f^{-1}$. Thus, if α, β and $\alpha+\beta$ are nonzero elements of F, then $\theta(\alpha) = \theta(1)\alpha^{-1}$, $\theta(\beta) = \theta(1)\beta^{-1}$ and

$$\theta(1)(\alpha^{-1} + \beta^{-1}) = \theta(\alpha) + \theta(\beta) = \theta(\alpha+\beta) = \theta(1)(\alpha+\beta)^{-1}.$$

Hence $\alpha^{-1} + \beta^{-1} = (\alpha+\beta)^{-1}$. In particular, if $x \neq 0, 1$ is in F, then setting $\alpha = x$ and $\beta = 1 - x$, we obtain $x^{-1} + (1-x)^{-1} = 1$ and hence x satisfies $x^2 - x + 1 = 0$. There are, of course, at most two solutions to the latter equation, so $|F| \leq 4$ contrary to our hypothesis.

Returning to the vector space $V = V_1 \oplus V_2$, we see that V_1 is a direct sum of G-submodules isomorphic to W_1, and V_2 is a direct sum of G-submodules isomorphic to W_2. Thus V is a completely reducible G-module and hence so is any submodule. Indeed, any submodule is a direct sum of copies of W_1 and of W_2. But $W_1 \not\cong W_2$, so any copy of W_1 in V is contained in V_1, and any copy of W_2 in V is contained in V_2. Thus, any G-submodule of V is of the form $A \oplus B$, as required. □

We actually showed above that if $W_1 \cong W_2$, then the inverse map in F extends to a field automorphism, and this does occur when $F = \mathrm{GF}(2)$, $\mathrm{GF}(3)$, or $\mathrm{GF}(4)$.

2. Finite Fields

In this section we obtain a few combinatorial results on finite fields. We start with a corollary to a simple special case of a result of [3]. We include the quick proof for the convenience of the reader.

Lemma 2.1. *Let E be a finite field, let $V \neq 0$ be an additive subgroup of E^+, and let $0 \neq s \in E$. If $sx^{-1} \in V$ for all $0 \neq x \in V$, then $V = Lt$ where L is a subfield of E and $t^2 \in Ls$.*

Proof. Let $x, y \in V$ with $xy \neq 0$ or s. Then $0 \neq (xy - s)/y = x - (s/y) \in V$, and hence $sy/(xy - s) \in V$. Thus $xy^2/(xy - s) = y + sy/(xy - s) \in V$ and, by taking inverses and multiplying by s, we have $s(xy - s)/xy^2 \in V$. It follows that $s^2/xy^2 = (s/y) - s(xy - s)/xy^2 \in V$ and therefore $xy^2/s = s(xy^2/s^2) \in V$. Of course, this inclusion is satisfied when $xy = 0$ or s, so we see that $xy^2/s \in V$ for all $x, y \in V$.

Now let $L = \{r \in E \mid Vr \subseteq V\}$. Then L is a subring of E and hence also a subfield, so V is an L-vector space. Notice that $xy^2/s \in V$ for all $x, y \in V$, so $y^2/s \in L$ and thus, since $0 \in V$, we have $|L| > |V|/2$. In particular, if $\dim_L V \geq 2$, then $|L| > |V|/2 \geq |L|^2/2$, so $2 > |L|$, a contradiction. Thus $\dim_L V = 1$ and $V = Lt$ for some $0 \neq t \in E$. Finally, since $s/t \in V = Lt$, we have $t^2 \in Ls$. □

Next, we extend the above argument to prove

Lemma 2.2. *Let $F \subseteq E$ be finite fields with $|E : F| \geq 3$, let $0 \neq s \in E$, and let $\lambda \colon E \to F$ be an F-linear functional with $\lambda(1) = 1$. Define the subset H of F by*

$$H = \{\lambda(sx^{-1}) \mid x \in E,\ \lambda(x) = 1\}.$$

Then $0 \in H$ and there exists a fixed $0 \neq b \in F$, such that every element $a \in F$ satisfies a polynomial equation of the form

$$a^2\tau b - ab + \sigma = 0$$

for suitable $\tau, \sigma \in H$ depending upon a. In particular, $|H| \geq \sqrt{|F|/2}$.

Proof. Let $|F| = q$ and $|E| = q^n$ with $n = |E : F|$. If $V = \ker \lambda$, then V is an F-subspace of E of dimension $n - 1$. Furthermore,

$$V + 1 = \{r \in E \mid \lambda(r) = 1\}.$$

Suppose $sv^{-1} \in V$ for all $0 \neq v \in V$. Then, by the preceding lemma, $V = Lt$ for some proper subfield L of E and some $0 \neq t \in E$. Indeed, since $FV \subseteq V$, we have $F \subseteq L$ and hence

$$n - 1 = \dim_F V = \dim_F L = |L : F| \leq n/2,$$

contradicting the assumption that $n \geq 3$.

Thus there exists an element $0 \neq v \in V$ with $s/v \notin V$. Then $0 \neq d = \lambda(s/v) \in F$, and if we set $y = s/vd$, then we have $\lambda(y) = 1$ and $s/y = vd \in V$. In particular, $0 = \lambda(sy^{-1}) \in H$. Note that $0 \notin V + 1$, so the map $V + 1 \to E$ given by $x \mapsto s^2/xy^2$ has image of size $|V|$ and this image does not contain 0. Thus, this image cannot be contained entirely within V. We now fix $x \in V + 1$ with $s^2/xy^2 \notin V$, and we set $0 \neq b = \lambda(s^2/xy^2)$. To reiterate, we have fixed $x, y \in V + 1$ with $s/y \in V$ and with $0 \neq b = \lambda(s^2/xy^2) \in F$.

Let $a \in F$ be arbitrary. Then $as/y \in V$, so

$$\frac{xy+as}{y} = x + \frac{as}{y} \in V+1$$

and we set $\tau = \tau_a = \lambda(sy/(xy+as)) \in F$, so that $\tau \in H$. For convenience, define

$$z = \frac{xy^2}{xy+as} = y - \frac{asy}{xy+as}$$

and let $c = c_a = \lambda(z)$. Since $\lambda(y) = 1$ and $\lambda(asy/(xy+as)) = a\tau$, we have $c = 1 - a\tau$. If $c = 0$, then a clearly satisfies

$$a^2\tau b - ab + \sigma = \sigma - cab = 0$$

with $0 = \sigma \in H$.

Now suppose that $c \neq 0$. Thus $z/c \in V+1$ and we let $\sigma = \sigma_a = \lambda(cs/z)$ so that $\sigma \in H$. Note that

$$\frac{cas^2}{xy^2} = \frac{cs(xy+as)}{xy^2} - \frac{cs}{y} = \frac{cs}{z} - \frac{cs}{y}$$

so $cab = ca\lambda(s^2/xy^2) = \lambda(cas^2/xy^2) = \sigma$, since $\lambda(cs/z) = \sigma$ and $\lambda(cs/y) = 0$. Also using $c = 1 - a\tau$, we have $(1-a\tau)ab = cab = \sigma$ and thus a does indeed satisfy the polynomial equation

$$a^2\tau b - ab + \sigma = 0$$

determined by σ and τ.

Write $h = |H|$ and note that $b \neq 0$. Then each pair $(\tau, \sigma) \in H \times H$ determines a unique nonzero polynomial of degree ≤ 2 given by

$$\zeta^2\tau b - \zeta b + \sigma \in F[\zeta]$$

and hence we have h^2 of these. Furthermore, each of these has at most two roots in F, so we obtain at most $2h^2$ roots in this mannner. On the other hand, we have shown above that each $a \in F$ is a root of at least one of these polynomials. Thus, $2h^2 \geq q$ and $h \geq \sqrt{q/2}$, as required. $\square$

We can sharpen the latter inequality a bit by considering separately those polynomials with either $\tau = 0$ or $\sigma = 0$ since they have at most one nonzero root.

We also need some results on linear functionals. Let V be a vector space over the finite field F of size $|F| = q$ and let $M = \{v_1, v_2, \ldots, v_m\}$ be a linearly independent subset of V of size m. If $\widehat{V}$ denotes the dual space of V, define

$$\mathrm{On}(M) = \{\lambda \in \widehat{V} \mid \lambda(M) = F\}$$

and

$$\mathrm{Non}(M) = \{\lambda \in \widehat{V} \mid \lambda(M) < F\}.$$

Thus $\mathrm{On}(M)$ is the set of linear functionals that map M onto F, and $\mathrm{Non}(M)$ is its complement in $\widehat{V}$. Obviously, $\mathrm{On}(M) = \emptyset$ if and only if $m < q$. In the following, we let ln denote the natural logarithm.

Lemma 2.3. *Let V be a vector space over F with $|F| = q$ and $\dim_F V = n$. In addition, let M be a linearly independent subset of V of size m and suppose that $m \geq 2q \ln q$. Then $|\mathrm{Non}(M)| < q^{n-1}$ and $|\mathrm{On}(M)| > q^n - q^{n-1}$. In particular, the latter two inequalities hold when $m \geq q^2$.*

Proof. Extend M to a basis of V by adding the vectors $w_1, w_2, \dots, w_k$ with $m+k = n$. We obtain a quick upper bound for $|\mathrm{Non}(M)|$ by using the first term of inclusion-exclusion. Specifically, note that there are q subsets $F_1, F_2, \dots, F_q$ of F of size $q-1$, and $\mathrm{Non}(M)$ is the union over i of all linear functionals λ with $\lambda(M) \subseteq F_i$. Now for each i and j, there are $q-1$ choices for $\lambda(v_j) \in F_i$ and q choices for $\lambda(w_j) \in F$. Thus the number of functionals with $\lambda(M) \subseteq F_i$ is precisely $(q-1)^m q^k$, and hence $|\mathrm{Non}(M)| \le q{\cdot}(q-1)^m q^k$.

For this result we want $|\mathrm{Non}(M)| < q^{n-1} = q^{m+k-1}$, and so it suffices to have $q{\cdot}(q-1)^m q^k < q^{m+k-1}$ or $(q-1)^m < q^{m-2}$. Taking logarithms, this is equivalent to $m \ln(q-1) < (m-2)\ln q$ or $2\ln q < m\big(\ln q - \ln(q-1)\big)$. Since $\ln q - \ln(q-1) > 1/q$, it therefore suffices to have $m/q \ge 2\ln q$ or $m \ge 2q\ln q$. Since $\mathrm{On}(M)$ is the complement of $\mathrm{Non}(M)$ in $\widehat{V}$, we see that $m \ge 2q \ln q$ implies that $|\mathrm{On}(M)| > q^n - q^{n-1}$. Finally, $q > 2\ln q$, so $m \ge q^2$ implies $m \ge 2q\ln q$. □

Of course, $|\mathrm{On}(M)|$ can be described precisely using full inclusion-exclusion. Its m-part, as given in the above proof, can also be written as $q!\, S(m,q)$, where $S(m,q)$ denotes the Stirling number of the second kind (see for example [2, pages 287 and 317]). We close with a well-known observation.

Lemma 2.4. *Let $E \supseteq F$ be fields with $|E:F| < \infty$ and let $\lambda\colon E \to F$ be a nonzero F-linear functional. Then every F-linear functional from E to F is uniquely of the form λ_a for $a \in E$, where $\lambda_a(x) = \lambda(ax)$.*

Proof. The map $a \mapsto \lambda_a$ is easily seen to be an F-linear transformation from E to $\widehat{E}$, and since E is a field, this map is one-to-one. By dimension considerations, the map is therefore also onto. □

3. G-Stable Ideals

We now begin our work on Theorem 1.1. In the following E, F and L will denote fields of characteristic $p > 0$. For the most part, they will be finite or at least locally finite. In particular, they will be subfields of a fixed algebraic closure of $\mathrm{GF}(p)$. Furthermore, any vector space over any of these fields is additively an elementary abelian p-group. In addition, we let K be a field of characteristic different from p, and we assume until further notice that K is algebraically closed or at least that it contains a primitive pth root of unity ε. We fix the prime p and the field K throughout. The following facts are standard.

Lemma 3.1. *Let V be a finite elementary abelian p-group and let G be a group of automorphisms of V.*

- *i. The group algebra $K[V]$ is semisimple. Indeed, it is a direct sum of $|V|$ copies of K and every ideal is uniquely an intersection of maximal ideals.*
- *ii. The maximal ideals of $K[V]$ are in one-to-one correspondence with the linear characters $\chi\colon V \to K^\bullet$. To be precise, the ideal corresponding to χ is the kernel of the natural algebra extension $\chi\colon K[V] \to K$.*
- *iii. G permutes the linear characters of V by defining $\chi^g(x) = \chi(x^{g^{-1}})$ for all $g \in G$ and $x \in V$. This action corresponds to the permutation action of G on the maximal ideals of $K[V]$.*
- *iv. Every G-stable ideal of $K[V]$ is uniquely an intersection of the maximal G-stable ideals of $K[V]$. The latter are precisely the intersections of G-orbits of maximal ideals of $K[V]$.*

v. $\chi^g = \chi$ *if and only if* $x^{-1}x^{g^{-1}} \in \ker\chi$ *for all* $x \in V$.

Now suppose F and $V = V_1 \oplus V_2$ are both finite. If $g \in G = F^\bullet$ fixes a character χ of V, then by (v) above, we have $\ker\chi \supseteq V_1(g-1) + V_2(g^{-1}-1)$, in additive notation. In particular, if $g \neq 1$, then $\ker\chi \supseteq V$ and χ is the trivial (i.e. principal) character of V. In other words, G permutes all the nontrivial characters of V in orbits of full size $|G|$.

Lemma 3.2. *Let F be a finite field with $|F| = q$ and let $G = F^\bullet$ act as the torus on $V = V_1 \oplus V_2$, where both V_1 and V_2 are 1-dimensional.*

- *i.* *$K[V_1]$ has precisely two maximal G-stable ideals, namely $\omega(V_1; V_1)$ and one other which we denote by J_1. They satisfy $\omega(V_1; V_1) \cap J_1 = 0$.*
- *ii.* *$K[V_2]$ has precisely two maximal G-stable ideals, namely $\omega(V_2; V_2)$ and one other which we denote by J_2. They satisfy $\omega(V_2; V_2) \cap J_2 = 0$.*
- *iii.* *There are $q+2$ maximal G-stable ideals of $K[V]$. One is $\omega(V; V)$, and two others J_1 and J_2 satisfy $J_1 \cap \omega(V; V) = \omega(V_1; V)$ and $J_2 \cap \omega(V; V) = \omega(V_2; V)$. For convenience, J_1 and J_2 are said to be quasi-augmentation ideals, while the remaining $q-1$ maximal G-stable ideals different from $\omega(V; V)$ are said to be standard.*

Proof. We know that G permutes the nontrivial characters of $K[V_1]$, $K[V_2]$ and $K[V]$ in orbits of full size $|G| = q-1$. In particular, since $|V_1| = q$, it follows that there are just two maximal G-stable ideals of $K[V_1]$, namely $\omega(V_1; V_1)$ and J_1. Clearly, $\omega(V_1; V_1) \cap J_1 = 0$, so (i) is proved, and (ii) follows similarly.

For (iii), we have $|V| = q^2$, so there are $1 + (q^2-1)/(q-1) = q+2$ G-orbits on the characters of V and hence on the maximal ideals of $K[V]$. The trivial character of course corresponds to the augmentation ideal $\omega(V; V)$. Since $\omega(V_1; V)$ is a G-stable ideal of codimension q, we see that there is a maximal G-stable ideal J_1 with $J_1 \cap \omega(V; V) = \omega(V_1; V)$. Similarly, there exists a maximal G-stable ideal J_2 with $J_2 \cap \omega(V; V) = \omega(V_2; V)$. We have accounted for 3 of the maximal G-stable ideals of $K[V]$, so there are $(q+2) - 3 = q-1$ remaining ideals which we consider to be standard. $\square$

While the above gives us a quick count on the G-stable ideals of $K[V]$, it does not really give us a good description of them. So we take a closer look at the action of $G = F^\bullet$ on $V = F^2$ and on $K[V]$.

Example 3.3. Let F be a finite field and let $V = F^2$. Then the torus action of $F^\bullet$ on F^2 is given by $(a, b)^f = (f^{-1}a, fb)$ for all $a, b \in F$ and $f \in F^\bullet$. Of course, $F^\bullet$ also acts on the group algebra $K[F^2]$ and our goal here is to describe the maximal $F^\bullet$-stable ideals of $K[F^2]$. As usual, we assume that $\operatorname{char} F = p > 0$, $\operatorname{char} K \neq p$ and that K contains ε, a primitive pth root of unity.

Let $\mathrm{GF}(p)$ denote the prime subfield of F and let $\mu\colon F \to \mathrm{GF}(p)$ be a nonzero $GF(p)$-linear functional. Then, by Lemma 2.4, all linear functionals from F to $\mathrm{GF}(p)$ are of the form $\mu_a\colon F \to \mathrm{GF}(p)$ where $a \in F$ and $\mu_a(x) = \mu(ax)$. Hence all characters $\chi\colon F \to K^\bullet$ are given by $\chi_a(x) = \varepsilon^{\mu_a(x)} = \varepsilon^{\mu(ax)}$. Furthermore, since the characters from F^2 to $K^\bullet$ are necessarily products, they are all of the form

$$\chi_{a,b}(x, y) = \varepsilon^{\mu(ax)+\mu(by)} = \varepsilon^{\mu(ax+by)}.$$

These in turn extend to K-algebra homomorphisms $\chi_{a,b}\colon K[F^2] \to K$ and their kernels $I_{a,b} = \ker\chi_{a,b}$ are precisely the set of maximal ideals of $K[F^2]$.

Now $F^\bullet$ permutes these characters by

$$\begin{aligned}\chi^g_{a,b}(x,y) &= \chi_{a,b}((x,y)^{g^{-1}}) = \chi_{a,b}(gx, g^{-1}y)\\ &= \varepsilon^{\mu(agx+bg^{-1}y)} = \chi_{ag,bg^{-1}}(x,y)\end{aligned}$$

for all $g \in F^\bullet$, and hence we have

$$I^g_{a,b} = I_{ag,bg^{-1}}.$$

The $F^\bullet$-orbits of these ideals are now easily seen to be

$$\begin{aligned}\mathcal{O}_{0,0} &= \{I_{0,0}\}\\ \mathcal{O}_{0,*} &= \{I_{0,b} \mid 0 \neq b \in F\}\\ \mathcal{O}_{*,0} &= \{I_{a,0} \mid 0 \neq a \in F\}\\ \mathcal{O}_{d} &= \{I_{a,b} \mid a,b \in F,\ ab = d\} \quad \text{for all } 0 \neq d \in F.\end{aligned}$$

Furthermore, the maximal $F^\bullet$-stable ideals of $K[F^2]$ are precisely the intersections of the kernels $I_{a,b}$ over all members of an orbit. Thus, we can denote the maximal $F^\bullet$-stable ideals of $K[F^2]$ by $I_{0,0}$, $I_{0,*}$, $I_{*,0}$, and I_d for all $0 \neq d \in F$, where the subscripts of course correspond to the orbit notation.

It is easy to see that $I_{0,0} = \omega(F^2; F^2)$ is the augmentation ideal of $K[F^2]$. Furthermore $I_{0,0} \cap I_{0,*} = \omega(F \oplus 0; F^2)$ and $I_{0,0} \cap I_{*,0} = \omega(0 \oplus F; F^2)$, where $F \oplus 0$ and $0 \oplus F$ are the obvious $F^\bullet$-stable subgroups of F^2. In other words, $I_{0,*}$ and $I_{*,0}$ are the quasi-augmentation maximal $F^\bullet$-stable ideals of $K[F^2]$, while the various I_d are the standard maximal $F^\bullet$-stable ideals. Of course, the intersection of all these maximal $F^\bullet$-stable ideals in 0.

Our goal now is to show that standard ideals do not appear in certain situations.

Lemma 3.4. *Let $E \supseteq L \supseteq F$ be finite fields with $|F| = q$. Suppose that $|E : L| \geq 3$, $|L : F| \geq 2q^2 + 1$, and let E^2 admit the torus action of $E^\bullet$. If I is a standard maximal $E^\bullet$-stable ideal of $K[E^2]$, then $I \cap K[F^2] = 0$.*

Proof. We use the description and notation for $K[F^2]$ as given in the preceding example. Of course, that example applies equally well to $K[E^2]$ and $K[L^2]$ provided we have appropriate functionals to the prime subfield $\mathrm{GF}(p)$. For this, we choose $\lambda\colon E \to L$, an L-linear functional with $\lambda(1) = 1$, and we choose $\eta\colon L \to F$, an F-linear functional with $\eta(1) = 1$. Then the composite functional $\bar\eta = \mu\eta$ maps L to $\mathrm{GF}(p)$, while $\bar\lambda = \mu\eta\lambda = \bar\eta\lambda$ maps E to $\mathrm{GF}(p)$.

Now we are given a standard maximal $E^\bullet$-stable ideal I of $K[E^2]$. Using the functional $\bar\lambda$, I is then equal to I_s for some $0 \neq s \in E$. In other words, I is the intersection of the kernels of the algebra homomorphisms $K[E^2] \to K$ associated with the characters

$$\chi_{a,b}(x,y) = \varepsilon^{\bar\lambda(ax+by)} = \varepsilon^{\bar\eta\lambda(ax+by)}$$

for all $a,b \in E$ with $ab = s$. Alternately, we can write $a = sz^{-1}$ and $b = z$ for all $0 \neq z \in E$.

Since $I_{a,b}$ is the kernel of algebra homomorphism $\chi_{a,b}\colon K[E^2] \to K$, it follows that $I_{a,b} \cap K[L^2]$ is the kernel of the restriction of $\chi_{a,b}$ to $K[L^2]$. Furthermore, since $\lambda\colon E \to L$ is an L-linear functional, we see that $\lambda(ax + by) = \lambda(a)x + \lambda(b)y$ for all $x, y \in L$. Thus, with $a = sz^{-1}$ and $b = z$, we see that the restriction of the character $\chi_{a,b}$ to L^2 is given by

$$\tilde\chi(x,y) = \varepsilon^{\bar\eta(\lambda(sz^{-1})x + \lambda(z)y)}.$$

In particular, if we consider only those z with $\lambda(z)=1$, then the various $\tilde{\chi}$ that are obtained in this manner correspond to $L^\bullet$-orbits with product given by $\lambda(sz^{-1})$. Since $|E:L|\geq 3$, Lemma 2.2 tells us that

$$H=\{\lambda(sz^{-1})\mid z\in E,\, \lambda(z)=1\}$$

satisfies $0\in H$ and $|H|\geq\sqrt{|L|/2}$.

To reiterate, if we use $\tilde{\chi}$ to denote the characters of L^2, then we have shown that the $L^\bullet$-stable ideal $I\cap K[L^2]$ is contained in the intersection of the kernel ideals corresponding to the $L^\bullet$-orbits of the characters

$$\tilde{\chi}_{h,1}=\varepsilon^{\bar{\eta}(hx+y)}$$

for all $h\in H$. Thus we have obtained a reasonably large number of $L^\bullet$-orbits in this restriction, but not enough to guarantee that $I\cap K[L^2]=0$.

So, we require a second pass, this time from L to F. If $m=q^2$, then by assumption, $n=|L:F|\geq 2m+1$. Thus $|L|/2\geq q^{2m+1}/2\geq q^{2m}$ and hence $|H|\geq\sqrt{|L|/2}\geq q^m$. It follows that the F-linear span of the elements of H has F-dimension at least m, and hence we can choose $h_1,h_2,\ldots,h_m\in H$ so that $M=\{h_1,h_2,\ldots,h_m\}$ is an F-linearly independent subset of L of size m. Since $m=q^2$, it now follows from Lemma 2.3 that

$$\mathrm{Nom}(M)=\{\tau\colon L\to F\mid \tau(M)<F\}$$

has size strictly less than q^{n-1}. Here, of course, each such τ is an F-linear functional.

On the other hand, note that $C=\{c\in L\mid \eta(c)=1\}$ is a coset of the kernel of η and hence has size q^{n-1}. Thus, since $0\notin C$, this set gives rise to precisely q^{n-1} distinct F-linear functionals $\tau\colon L\to F$ by taking $\tau(x)=\eta_{c^{-1}}(x)=\eta(c^{-1}x)$ for all $c\in C$. But $|\mathrm{Nom}(M)|<q^{n-1}$, so there exists $d\in C$ with $\eta(d^{-1}M)=F$ and hence with $\eta(d^{-1}H)=F$. Of course, $d\in C$ says that $\eta(d)=1$.

Now, we know that $\tilde{I}=I\cap K[L^2]$ is an $L^\bullet$-stable ideal of $K[L^2]$ that is contained in the maximal $L^\bullet$-stable ideals corresponding to at least the characters $\tilde{\chi}_{h,1}$ with $h\in H$. Thus $\tilde{I}$ is contained in the kernel of the algebra homomorphisms corresponding to the characters

$$\tilde{\chi}_{hd^{-1},d}(x,y)=\varepsilon^{\mu\eta(hd^{-1}x+dy)}$$

for all $h\in H$. Since η is an F-linear functional with $\eta(d)=1$, we have

$$\eta(hd^{-1}x+dy)=\eta(hd^{-1})x+\eta(d)y=\eta(hd^{-1})x+y$$

for all $x,y\in F$. Thus the restriction of $\tilde{\chi}_{hd^{-1},d}$ to F^2 is given by

$$\bar{\chi}(x,y)=\varepsilon^{\mu(\eta(hd^{-1})x+y)}.$$

Since $\eta(Hd^{-1})=F$, we see that $I\cap K[F^2]=\tilde{I}\cap K[F^2]$ is an $F^\bullet$-stable ideal of $K[F^2]$ contained in the kernel ideals associated to all orbits except possibly $\mathcal{O}_{0,0}$ and $\mathcal{O}_{*,0}$. But the situation here is really right-left symmetric, and we know that $I\cap K[F^2]$ is contained in the ideal corresponding to $\mathcal{O}_{0,*}$, so it must also be contained in the ideal corresponding to $\mathcal{O}_{*,0}$. This leaves only $\omega(F^2;F^2)$, the ideal corresponding to $\mathcal{O}_{0,0}$. For this, note that $0\in H$, so $\tilde{I}$ is contained in the ideal corresponding to the orbit $\tilde{\chi}_{0,t}$ for all $0\neq t\in L$. We can, of course, choose a suitable nonzero element t with $\eta(t)=0$, and hence the restriction of this character to F^2 is trivial. We conclude that $I\cap K[F^2]$ is contained in all maximal $F^\bullet$-stable ideals of $K[F^2]$, and consequently $I\cap K[F^2]=0$, as required. $\square$

As an immediate consequence, we have

Lemma 3.5. *Let $E \supseteq L \supseteq F$ be finite fields with $|F| = q$. Suppose that $|E : L| \geq 3$, $|L : F| \geq 2q^2 + 1$, and let E^2 admit the torus action of $E^\bullet$. If I is an $E^\bullet$-stable ideal of $K[E^2]$ with $I \subseteq \omega(E^2; E^2)$ and $I \cap K[F^2] \neq 0$, then $I = \omega(E^2; E^2)$ or $\omega(E \oplus 0; E^2)$ or $\omega(0 \oplus E; E^2)$ or $\omega(E \oplus 0; E^2) \cap \omega(0 \oplus E; E^2)$.*

Proof. We know that I is an intersection of certain maximal $E^\bullet$-stable ideals of $K[E^2]$, and let J be one of these ideals. If J is standard, then the preceding lemma implies that $J \cap K[F^2] = 0$ and hence $I \cap K[F^2] = 0$, a contradiction. Thus J can only be one of the two quasi-augmentation ideals J_1 and J_2, and also $\omega(E^2; E^2)$, which we know does occur by assumption. Since $J_1 \cap \omega(E^2; E^2) = \omega(E \oplus 0; E^2)$ and $J_2 \cap \omega(E^2; E^2) = \omega(0 \oplus E; E^2)$, the result follows. □

It is now a simple matter to prove the main result of this paper. We will use the general machinery developed in [7, Section 1] even though some of this machinery could be fairly easily avoided in the present context.

Proof of Theorem 1.1. Let F be an infinite locally finite field of characteristic $p > 0$ and let $G = F^\bullet$ act on $V = F^2$ as the torus. Then G acts on the group algebra $K[V]$ with char $K \neq p$, and we begin by assuming that K contains a primitive pth root of unity. We first show that if I is a nonzero G-stable ideal of $K[V]$ with $I \subseteq \omega(V; V)$, then I is a finite intersection of the augmentation ideals corresponding to certain G-stable subgroups of V.

To this end, since $I \neq 0$, there exists a finite subfield F_0 of F with $I \cap K[F_0^2] \neq 0$. If $|F_0| = q$, choose a finite subfield L_0 of F with $L_0 \supseteq F_0$ and with $|L_0 : F_0| \geq 2q^2+1$. Next let E_0 be a finite subfield of F with $E_0 \supseteq L_0$ and $|E_0 : L_0| \geq 3$. Since F is countably infinite and locally finite, we can now find a chain of finite subfields $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ of F with $F = \bigcup_i E_i$. Notice that $\{(E_i^2, E_i^\bullet) \mid i = 0, 1, 2, \ldots\}$ is a local system for (V, G) in the notation of [7, Section 1]. Furthermore, for each i, the preceding lemma, applied to the fields $E_i \supseteq L_0 \supseteq F_0$, implies that $I \cap K[E_i^2]$ is an intersection of augmentation ideals of $E_i^\bullet$-stable subgroups of E_i^2. Thus, by [7, Lemma 1.2], I is an intersection (possibly infinite) of augmentation ideals corresponding to G-stable subgroups of V. But, by Lemma 1.2, there are only three nonidentity G-stable subgroups of V, namely V, $F \oplus 0$ and $0 \oplus F$, so this intersection must be finite.

Next, we show that any G-stable ideal of $K[F^2]$ is contained in $\omega(V; V)$ and hence has the above form. To this end, observe that Lemma 1.2 implies that all G-sections of V are infinite. Hence by [7, Lemma 1.3], if A is any G-stable subgroup of V, then $\omega(A; V)$ is a G-prime ideal of $K[V]$. In particular, $K[V]$ itself is G-prime. Now let I be a G-stable ideal of $K[V]$ and assume that $I \neq 0$. Then, by G-primeness, $J = I \cdot \omega(V; V)$ is a nonzero G-stable ideal contained in $\omega(V; V)$. If $J = \omega(V; V)$, then $I \supseteq J$ implies that $I = \omega(V; V)$. Otherwise, by the result of the previous paragraph, $J \subseteq \omega(A; V)$ for some G-stable subgroup A properly smaller than V. But $\omega(A; V)$ is a G-prime ideal and $I \cdot \omega(V; V) \subseteq \omega(A; V)$, so we conclude that $I \subseteq \omega(A; V) \subseteq \omega(V; V)$, as required.

We now know that any nonzero G-stable ideal of $K[V]$ is contained in $\omega(V; V)$ and hence is one of four possibilities, namely $\omega(V; V)$, $\omega(V_1; V)$, $\omega(V_2; V)$ and $\omega(V_1; V) \cap \omega(V_2; V)$, where $V_1 = F \oplus 0$ and $V_2 = 0 \oplus F$. In other words, the theorem is proved when K contains a primitive pth root of unity.

It remains to consider arbitrary fields K with $\operatorname{char} K \neq p$. In this situation, we let $\overline{K}$ be an extension of K that contains a primitive pth root of unity. If I is a nonzero G-stable ideal of $K[V]$, then $\overline{K}\cdot I$ is a nonzero G-stable ideal of $\overline{K}[V]$, and the freeness of $\overline{K}$ over K easily implies that $(\overline{K}\cdot I) \cap K[V] = I$. Since $\overline{K}\cdot I$ is an intersection of augmentation ideals $\omega_{\overline{K}}(A;V)$ and since $\omega_{\overline{K}}(A;V) \cap K[V] = \omega(A;V)$, the result follows. $\square$

We remark that the same techniques can be used to handle the case of the torus G of $\mathrm{SL}_n(F)$ acting on F^n for any integer n. The result one obtains is then quite analogous to that of Theorem 1.1, and a full proof will appear elsewhere.

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