INVARIANT IDEALS OF ABELIAN GROUP ALGEBRAS UNDER THE TORUS ACTION OF A FIELD, II

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ABSTRACT. Let $V = V_1 \oplus V_2$ be a finite-dimensional vector space over an infinite locally-finite field F. Then V admits the torus action of $G = F^\bullet$ by defining $(v_1 \oplus v_2)^g = v_1 g^{-1} \oplus v_2 g$. If K is a field of characteristic different from that of F, then G acts on the group algebra $K[V]$ and it is an interesting problem to determine all G-stable ideals of this algebra. In this paper, we show that, for almost all fields F, the G-stable ideals are uniquely writable as finite irredundant intersections of augmentation ideals of subspaces $W_1 \oplus W_2$, with $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$. As a consequence, the set of all G-stable ideals is Noetherian.

1. Introduction

Let K be a field, let V be an abelian group, and let G be a group of automorphisms of V. Then G acts on the group algebra $K[V]$ and it is an interesting, and surprisingly difficult, problem to describe the G-stable ideals of $K[V]$. The motivation for this actually comes from the study of the lattice of ideals in group algebras of certain infinite locally finite groups. This can be seen, for example, in the survey [4] or in the introduction to paper [6], but we will not expand upon this theme here. A natural special case of the problem occurs when V is a vector space over an infinite field F and when $G = F^\bullet$ acts on V by scalar multiplication. This turns out to have a rather beautiful solution [1, 6, 2], especially when V is finite dimensional. Indeed, one can even allow F to be a division algebra.

The next case of interest surely arises by introducing inverses from F. Specifically, let V_1 and V_2 be two vector spaces over F, form $V = V_1 \oplus V_2$, and let $G = F^\bullet$ act on V by $(v_1 \oplus v_2)^g = v_1 g^{-1} \oplus v_2 g$. We call this the *torus action* of F by analogy to the way the torus in $\mathrm{SL}_2(F)$ acts on F^2. The goal of this paper is to describe the G-stable ideals of $K[V]$ when F is an infinite locally finite field and V is a finite-dimensional F-vector space. The argument here is a continuation of the proof given in paper [5], where the special case of $\dim_F V_1 = 1 = \dim_F V_2$ was considered.

Unfortunately, in this paper, we require an additional assumption on F, namely that of wideness. Specifically, let $\overline{\mathrm{GF}(p)}$ denote the algebraic closure of $\mathrm{GF}(p)$ and let $\mathrm{GF}(p) \subseteq F \subseteq \overline{\mathrm{GF}(p)}$. We note that the minimal (nonprime) subfields of $\overline{\mathrm{GF}(p)}$ are precisely the fields $\mathrm{GF}(p^r)$, where $r > 1$ is prime, and we say that F is *wide* if it contains infinitely many of these minimal subfields. Otherwise, of course, F is *narrow*. Now it is clear that $\overline{\mathrm{GF}(p)}$ has uncountably many subfields, but only countably many of these are narrow. Thus the wide subfields account for almost

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2000 *Mathematics Subject Classification.* 16S34, 12E20.

Research supported in part by NSA grant H98230-10-1-0217.

all of the subfields of $\overline{\mathrm{GF}(p)}$. At present, we do not know whether the wideness assumption is necessary for the theorem or just for the proof we offer here.

Now let V be an abelian group, viewed multiplicatively, and let $K[V]$ denote its group algebra over the field K. If A is a subgroup of V, then there exists a natural epimorphism $K[V] \to K[V/A]$ and we let $\omega(A;V) = \omega_K(A;V)$, the *augmentation ideal* of A in V, denote its kernel. Thus, $\omega(A;V)$ is the K-linear span of all elements of the form $(1-a)v$ with $a \in A$ and $v \in V$, and clearly

$$A = \{v \in V \mid 1 - v \in \omega(A;V)\}.$$

Observe that if A and B are subgroups of V and if $C = \langle A, B\rangle$ is the group they generate, then $\omega(A;V) + \omega(B;V) = \omega(C;V)$. Indeed, if I denotes the ideal $\omega(A;V) + \omega(B;V)$, then surely $I \subseteq \omega(C;V)$. On the other hand, both A and B are contained in the kernel of the homomorphism $K[V] \to K[V]/I$ restricted to the group V, and hence C is also contained in this kernel. Now, if G is a group that acts as automorphisms on V, then G also acts on $K[V]$, and it is clear that A is a G-stable subgroup of V if and only if $\omega(A;V)$ is a G-stable ideal of $K[V]$.

We now return to additive notation for V. The main result of this paper is

Theorem 1.1. *Let F be an infinite locally-finite field, let V_1 and V_2 be two finite-dimensional F-vector spaces, and set $V = V_1 \oplus V_2$. If $G = F^\bullet$, then we can let G act as the torus on V, and hence G acts on the group algebra $K[V]$. Suppose, in addition, that* $\operatorname{char} K \neq \operatorname{char} F$ *and that F is wide. Then every G-stable ideal of $K[V]$ is uniquely a finite irredundant intersection of augmentation ideals $\omega(W_1 \oplus W_2; V)$ with W_1 an F-subspace of V_1 and W_2 an F-subspace of V_2. As a consequence, the set of G-stable ideals of $K[V]$ is Noetherian.*

Note that, if V is a torsion abelian group having no elements of order equal to the characteristic of K, then $K[V]$ is a commutative von Neumann regular algebra (see [3, Theorem 1.1.5]). It follows that if $I, J \lhd K[V]$, then $I \cap J = IJ$. In particular, finite products and finite intersections of ideals coincide here. Furthermore, every ideal of $K[V]$ is semiprime.

We remark that the G-stable subgroups of $V = V_1 \oplus V_2$ were described in [5, Lemma 1.2]. Specifically, we have

Lemma 1.2. *Let F be a field that is not necessarily locally finite and assume that $|F| \geq 5$. If $V = V_1 \oplus V_2$ is an F-vector space admitting the torus action of $G = F^\bullet$, then the G-stable subgroups of V are precisely those of the form $W_1 \oplus W_2$, where W_1 is an F-subspace of V_1 and W_2 is an F-subspace of V_2.*

2. Functionals

For the most part, we will be concerned here with finite fields $E \supseteq F$ and with the space $\widehat{E}$ of F-linear functionals $\lambda\colon E \to F$. These functionals are related to each other by the following well-known

Lemma 2.1. *Let $E \supseteq F$ be fields with $|E : F| < \infty$ and let $\lambda\colon E \to F$ be a nonzero F-linear functional. Then every F-linear functional from E to F is uniquely of the form λ_a for $a \in E$, where $\lambda_a(x) = \lambda(ax)$.*

Proof. The map $a \mapsto \lambda_a$ is easily seen to be an F-linear transformation from E to $\widehat{E}$, and since E is a field, this map is one-to-one. By dimension considerations, the map is therefore also onto. $\square$

In the case of finite fields, E/F is always Galois, so we can certainly choose $\lambda\colon E \to F$ to be the Galois trace. Specifically, if $|F| = q$ and $|E : F| = n$, then this trace is given by

$$\mathrm{tr}_{E/F}(x) = \sum_{i=0}^{n-1} x^{q^i}.$$

This is a particularly convenient functional since, for $E \supseteq L \supseteq F$, we have

$$\mathrm{tr}_{E/F} = \mathrm{tr}_{L/F} \circ \mathrm{tr}_{E/L}\,.$$

Note that some of the arguments in [5] use functional λ with $\lambda(1) = 1$. So, if we want to be able to quote these results and still use Galois traces, then we have to avoid extensions $E \supseteq F$ with degree $|E : F| = n$ divisible by the characteristic of the field. If we do this, then we can define the normalized trace $\overline{tr}$ by

$$\overline{tr}_{E/F}(x) = \frac{1}{n}\cdot \mathrm{tr}_{E/F}(x) = \frac{1}{n}\cdot \sum_{i=0}^{n-1} x^{q^i}.$$

Since the degree is multiplicative, we also have

$$\overline{tr}_{E/F} = \overline{tr}_{L/F} \circ \overline{tr}_{E/L}$$

when $E \supseteq L \supseteq F$.

We remark that a version of this transitivity holds for any linear functional. Indeed, we have

Lemma 2.2. *Let $E \supseteq L \supseteq F$ be fields with $|E : F| < \infty$ and let $\lambda\colon E \to F$ and $\eta\colon L \to F$ be F-linear functionals with $\eta \neq 0$. Then there exists an L-linear functional $\mu\colon E \to L$ such that $\lambda = \eta \circ \mu$.*

Proof. If $\bar{\mu}\colon E \to L$ is any nonzero L-linear functional, then $\bar{\lambda} = \eta \circ \bar{\mu}$ is a nonzero F-linear functional $\bar{\lambda}\colon E \to F$. Therefore the previous lemma implies that $\lambda = \bar{\lambda}_a$ for some $a \in E$. Since $\lambda(x) = \bar{\lambda}_a(x) = \eta(\bar{\mu}(ax)) = \eta(\bar{\mu}_a(x))$, we conclude that $\lambda = \bar{\lambda}_a = \eta \circ \bar{\mu}_a$, as required. □

For the remainder of this section, we consider only finite fields. The content of the next lemma is really contained in part (iii), since the first two parts are essentially obvious.

Lemma 2.3. *Let $|E : F| \geq 3$, and let $\lambda\colon E \to F$ be a nonzero F-linear functional. If $0 \neq a, b \in E$, then*

i. $\{x \in E \mid \lambda(ax) = 0\}$ *has size* $|E|/|F|$.

ii. $\{x \in E \mid \lambda(ax) = \lambda(bx) = 0\}$ *has size* $|E|/|F|^2$ *if* $ba^{-1} \notin F$.

iii. $\{x \in E^\bullet \mid \lambda(ax) = \lambda(bx^{-1}) = 0\}$ *has size* $< 2\cdot|E|/|F|^2$.

Proof. For convenience, write $|F| = q$ and $|E : F| = n \geq 3$, so that $|E| = q^n$. Set $V = \ker\lambda$. Then V is an F-subspace of E of codimension 1, and $\lambda(ax) = 0$ if and only if $x \in a^{-1}V$. Thus (i) follows since $|a^{-1}V| = |V| = q^{n-1} = |E|/|F|$.

For (ii), note that $\lambda(ax) = \lambda(bx) = 0$ if and only if $x \in a^{-1}V \cap b^{-1}V$, where the latter is an intersection of two subspaces of codimension 1. If $a^{-1}V = b^{-1}V$, then $(ba^{-1})V = V$ and V is a vector space over the intermediate field $L = F[ba^{-1}] \subseteq E$. But then E/V is also a vector space over L, and since E/V is 1-dimensional over F, we must have $L = F$ and $ba^{-1} \in F$. In other words, if $ba^{-1} \notin F$, then $a^{-1}V \neq b^{-1}V$, so $a^{-1}V \cap b^{-1}V$ is an F subspace of E of codimension 2, and hence $|a^{-1}V \cap b^{-1}V| = q^{n-2} = |E|/|F|^2$.

Finally, we consider (iii). In view of Lemma 2.1, if we change λ to a different nonzero functional, then we obtain the same problem, but with different nonzero parameters a and b. Thus it suffices to assume here that $\lambda\colon E \to F$ is the Galois trace. In particular, if $\lambda(ax) = 0$, then

$$(ax) + (ax)^q + \cdots + (ax)^{q^{n-2}} + (ax)^{q^{n-1}} = 0$$

and solving for $x^{q^{n-1}}$, we see that $x^{q^{n-1}} = f_a(x)$, where $f_a(x)$ is a polynomial in x, depending upon a, and having degree precisely q^{n-2}.

Next, if $\lambda(bx^{-1}) = 0$, then

$$(bx^{-1}) + (bx^{-1})^q + \cdots + (bx^{-1})^{q^{n-2}} + (bx^{-1})^{q^{n-1}} = 0,$$

and multiplying by $x^{q^{n-1}+q^{n-2}}$ yields

$$\begin{aligned} 0 &= b^{q^{n-1}} x^{q^{n-2}} + x^{q^{n-1}} \sum_{i=0}^{n-2} b^{q^i} x^{q^{n-2}-q^i} \\ &= b^{q^{n-1}} x^{q^{n-2}} + x^{q^{n-1}} g_b(x), \end{aligned}$$

where $g_b(x)$ is a polynomial in x, depending upon b, and having degree precisely $q^{n-2} - 1 \geq q - 1 \geq 1$. In other words, $n \geq 3$ implies that $\deg g_b(x) > 0$.

In particular, if $\lambda(ax) = 0$ and $\lambda(bx^{-1}) = 0$, then

$$0 = b^{q^{n-1}} x^{q^{n-2}} + x^{q^{n-1}} g_b(x) = b^{q^{n-1}} x^{q^{n-2}} + f_a(x) g_b(x) = h_{a,b}(x),$$

where $h_{a,b}(x)$ is a polynomial in x, depending upon a and b, and having degree precisely $q^{n-2} + q^{n-2} - 1 = 2q^{n-2} - 1 > q^{n-2}$. Of course, the polynomial $h_{a,b}(x)$ has at most $2q^{n-2} - 1$ roots in E. □

Part (iii) above is not true in general when $|E : F| = 2$ and the proof fails because the first term in $h_{a,b}(x)$ can cancel with the the product polynomial $f_a(x)g_b(x)$ to force $h_{a,b}(x)$ to be identically 0. Indeed, this occurs, for example, when $a = b = 1$. We conclude this section with the following rather technical consequences of the previous lemma.

Lemma 2.4. *Let $E \supseteq L \supseteq F$ be finite fields and let $\lambda\colon E \to L$ be a nonzero L-linear functional. Let $a, b \in E$ and $A, B \subseteq E$. Assume that*

i. $0 \neq A$ is an F-subspace of E of dimension $\leq r$, and $a \notin A$.
ii. B is an F-subspace of E of dimension $\leq s$, and $b \notin B$.
iii. $|E : L| \geq 3$ and $|L : F| \geq r + s + 2$.

If $a \notin LA$, then there exists $x \in E^\bullet$ such that $\lambda(ax) \notin \lambda(Ax)$, $\lambda(bx^{-1}) \notin \lambda(Bx^{-1})$, and $\dim_F \lambda(Ax) < \dim_F A$.

Proof. Choose $0 \neq \alpha_0 \in A$ and let $U = U(\alpha_0) = \{x \in E \mid \lambda(\alpha_0 x) = 0\}$. Then the previous lemma implies that $|U(\alpha_0)| = |E|/|L|$. Next, for each $\alpha \in A$, let $V(\alpha) = \{x \in E \mid \lambda(\alpha_0 x) = \lambda((a-\alpha)x) = 0\}$. Since $a \notin LA$, we have $(a-\alpha)/\alpha_0 \notin L$ and hence, by the previous lemma again, $|V(\alpha)| = |E|/|L|^2$. Finally, for each $\beta \in B$, let $W(\beta) = \{x \in E^\bullet \mid \lambda(\alpha_0 x) = \lambda((b-\beta)x^{-1}) = 0\}$. Since $b \notin B$, we have $b-\beta \neq 0$, so Lemma 2.3 and $|E : L| \geq 3$ imply that $|W(\beta)| < 2|E|/|L|^2$.

Now set $V = \bigcup_{\alpha\in A} V(\alpha)$ and $W = \bigcup_{\beta\in B} W(\beta)$. Then, by definition, $V, W \subseteq U$ and our next goal is to show that $U > V \cup W$. For this, note that $|U| = |E|/|L|$,

$|V| \le |A|{\cdot}|E|/|L|^2$ and $|W| < 2\,|B|{\cdot}|E|/|L|^2$. Thus, it suffices to verify the leftmost inequality in

$$|U| = \frac{|E|}{|L|} \ge |A|{\cdot}\frac{|E|}{|L|^2} + 2\,|B|{\cdot}\frac{|E|}{|L|^2} > |V| + |W| \ge |V \cup W|.$$

In other words, multiplying by $|L|^2/|E|$, we need

$$|L| \ge |A| + 2\,|B|.$$

Let us write $|F| = q$. Then using $\dim_F A \le r$, we have $|A| \le q^r$, and similarly $|B| \le q^s$. In particular, if $|L : F| = m$, then we need to show that

$$q^m \ge q^r + 2q^s,$$

and certainly the assumption $m \ge r+s+2$ suffices for this because of the inequality $(q^{r+1}-1)(q^{s+1}-1) \ge 1$.

We can now choose $x \in U \setminus (V \cup W) \subseteq E$, and notice that $x \in U$ implies that $\lambda(\alpha_0 x) = 0$. Since the map $A \to \lambda(xA)$ given by $\alpha \mapsto \lambda(x\alpha)$ is an F-linear transformation and since $0 \ne \alpha_0$ is in its kernel, we conclude that $\dim_F A > \dim_F \lambda(xA)$. Next, since $x \notin V$, we see that $\lambda((a-\alpha)x) \ne 0$ for all $\alpha \in A$ and hence $\lambda(ax) \notin \lambda(Ax)$. In particular, $x \ne 0$. Finally, since $x \notin W$, we have $\lambda((b-\beta)x^{-1}) \ne 0$ for all $\beta \in B$, and hence $\lambda(bx^{-1}) \notin \lambda(Bx^{-1})$. □

The previous result is essentially symmetric in A and B. Specifically, if we interchange the a- and b-terms and replace x by x^{-1}, then we obtain

Lemma 2.5. *Let $E \supseteq L \supseteq F$ be finite fields and let $\lambda\colon E \to L$ be a nonzero L-linear functional. Let $a, b \in E$ and $A, B \subseteq E$. Assume that*

i. A is an F-subspace of E of dimension $\le r$, and $a \notin A$.
ii. $0 \ne B$ is an F-subspace of E of dimension $\le s$, and $b \notin B$.
iii. $|E : L| \ge 3$ and $|L : F| \ge r + s + 2$.

If $b \notin LB$, then there exists $x \in E^\bullet$ such that $\lambda(ax) \notin \lambda(Ax)$, $\lambda(bx^{-1}) \notin \lambda(Bx^{-1})$, and $\dim_F \lambda(Bx^{-1}) < \dim_F B$.

In some sense, the weakness of the above two results is the additional hypothesis that $a \notin LA$ or that $b \notin LB$. We overcome this difficulty in the next section by introducing a special type of field extension E/F.

3. Square-Free Extensions

In this section, we consider only finite fields. If $E \supseteq F$ are two such fields, we say that the extension E/F is *square-free* if the degree $|E : F| = n$ is a square-free number, that is a product of distinct primes. In this case, we let $w(E/F)$, the *width* of the extension, denote the number of prime factors of n. It is clear that this width measures the number of intermediate fields that are minimal over F and also the number of intermediate fields that are maximal in E. Furthermore,

$$|E : F| \ge 2^{w(E/F)} \ge w(E/F) + 1.$$

If E/F is square-free and if L is an intermediate field, then the multiplicative nature of the degree implies that E/L and L/F are both square-free. In addition, we have $w(E/L) + w(L/F) = w(E/F)$. Square-free extensions E/F with $E \ne F$ have the semisimple-like property that F is the intersection of all maximal intermediate fields. We use this simple observation to prove

Lemma 3.1. *Suppose E/F is a square-free extension, let $\alpha_0, \alpha_1, \ldots, \alpha_m \in E$, and assume that these elements are linearly independent over F. Then there exists an intermediate field L with $w(E/L) \leq m$ and with $\alpha_0, \alpha_1, \ldots, \alpha_m$ linearly independent over L.*

Proof. We proceed by induction on m. If $m = 0$, then $\alpha_0 \neq 0$, so α_0 is independent over E. Hence, we can take $L = E$ and note that $w(E/L) = 0$. Now let $m \geq 1$. Then by induction, there exists an intermediate field C with $\alpha_1, \alpha_2, \ldots, \alpha_m$ linearly independent over C, and with $w(E/C) \leq m - 1$. If $\alpha_0, \alpha_1, \ldots, \alpha_m$ are linearly independent over C, we take $L = C$ and we are done. If not, we can write

$$\alpha_0 = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m \tag{$\dagger$}$$

for suitable $c_i \in C$. Indeed, since $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent over C, this is the unique linear expression for α_0 in terms of the other α_i's and having coefficients in C.

By assumption, $\alpha_0, \alpha_1, \ldots, \alpha_m$ are linearly independent over F. Thus we cannot have all c_i in F. Say $c_1 \in C \setminus F$. Then, since C/F is square-free, the semisimplicity property implies that there exists a maximal intermediate field L with $C > L \supseteq F$ and with $c_1 \notin L$. Obviously, $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent over L since $L \subseteq C$. Furthermore, we cannot write α_0 as an L-linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_m$ since ($\dagger$) is the unique such description with coefficients in C, and since $L \subseteq C$. Thus $\alpha_0, \alpha_1, \ldots, \alpha_m$ are L-linearly independent. Finally, we have $w(C/L) = 1$ and $w(E/C) \leq m-1$, so we conclude that $w(E/L) \leq (m-1)+1 = m$, and the result follows. $\square$

An interesting consequence is the following generalization.

Lemma 3.2. *Suppose E/F is a square-free field extension, let $a_0, a_1, \ldots, a_r$ be elements of E, and assume that we are given a F-linear functional $\mu\colon E \to F$. Then there exists an intermediate field L with $w(E/L) \leq r$ and an L-linear functional $\lambda\colon E \to L$ such that $\lambda(a_i) = \mu(a_i)$ for all i.*

Proof. If $\mu(a_i) = 0$ for all i, take $L = E$ and let $\lambda\colon E \to E$ be the zero map. We can now assume that some $\mu(a_i)$ is not zero, say $\mu(a_0) \neq 0$. Furthermore, by multiplying by a nonzero scalar in F if necessary, it clearly suffices to assume that $\mu(a_0) = 1$. Hence $\mu(v_i) = 0$ where $v_i = a_i - \mu(a_i)a_0$. Since $v_0 = 0$, the F-vector space V spanned by all v_i has dimension at most r, so we can let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be an F-basis for V with $m \leq r$. Note that $\mu(V) = 0$ but $\mu(a_0) = 1$, so $\alpha_0, \alpha_1, \ldots, \alpha_m$ is F-linearly independent, where we set $\alpha_0 = a_0$.

The preceding lemma now implies that there exists an intermediate field L with $w(E/L) \leq m \leq r$ such that the elements $\alpha_0, \alpha_1, \ldots, \alpha_m$ are L-linearly independent. In particular, we can assign values for an L-linear functional in an arbitrary manner on this independent set. In other words, there exists $\lambda\colon E \to L$ such that $\lambda(\alpha_0) = 1$ and $\lambda(\alpha_i) = 0$ for all $i > 0$. It follows that $\lambda(V) = 0$ and that $\lambda(a_0) = \lambda(\alpha_0) = 1$. Finally, for all $i \geq 0$, we have $0 = \lambda(v_i) = \lambda(a_i - \mu(a_i)a_0) = \lambda(a_i) - \mu(a_i)\lambda(a_0) = \lambda(a_i) - \mu(a_i)$, as required. $\square$

We now combine the ideas here with Lemmas 2.4 and 2.5 to prove

Lemma 3.3. *Let $E \supseteq F$ be finite fields with E/F a square-free extension of width $w(E/F) \geq (r+s+1)^2$, for suitable integers $r, s \geq 0$. Let $a, b \in E$ and let $A, B \subseteq E$, where A and B are F-subspaces of E. Assume that $\dim_F A \leq r$, $\dim_F B \leq s$, $a \notin A$*

and $b \notin B$. If $0 \le t \le r+s$, then there exists an intermediate field L and an element $x \in E^\bullet$ such that

i. $w(E/L) \le (r+s+1)t$.
ii. $\mathrm{tr}_{E/L}(ax) \notin \mathrm{tr}_{E/L}(Ax)$.
iii. $\mathrm{tr}_{E/L}(bx^{-1}) \notin \mathrm{tr}_{E/L}(Bx^{-1})$.
iv. $\dim_F \mathrm{tr}_{E/L}(Ax) + \dim_F \mathrm{tr}_{E/L}(Bx^{-1}) \le r+s-t$.

Proof. We proceed by induction on t, from $t=0$ to $t=r+s$. If $t=0$, we can of course take $E=L$ and $x=1$.

Now let use assume the result holds for some $t<r+s$, and let L and x be the solutions for this t. For convenience, write $\eta = \mathrm{tr}_{E/L}$. Then $w(E/L) \le (r+s+1)t$, so L/F is a square-free extension with $w(L/F) \ge (r+s+1)(r+s+1-t) \ge 2$. Also, if $\bar{a} = \eta(ax)$, $\bar{b} = \eta(bx^{-1})$, $\overline{A} = \eta(Ax)$ and $\overline{B} = \eta(Bx^{-1})$, then we know that $\bar{a} \notin \overline{A}$, $\bar{b} \notin \overline{B}$, $\dim_F \overline{A} \le \dim_F A \le r$, $\dim_F \overline{B} \le \dim_F B \le s$, and

$$\dim_F \overline{A} + \dim_F \overline{B} \le r+s-t.$$

It remains to consider the parameter $t+1$. If $\overline{A} = \overline{B} = 0$, then we can certainly take the same L and x for the solution to the $t+1$ case. Thus, it suffices to assume that at least one of $\overline{A}$ or $\overline{B}$ is not 0. By symmetry, say $\overline{A} \ne 0$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be an F-basis for $\overline{A}$, so that $1 \le k \le r$. Since $\bar{a} \notin \overline{A}$, we see that $\bar{a}, \alpha_1, \alpha_2, \ldots, \alpha_k$ are F-linearly independent elements of L. Thus, by Lemma 3.1, there exists an intermediate field L_0, with $w(L/L_0) \le k \le r$ and with $\bar{a}, \alpha_1, \alpha_2, \ldots, \alpha_k$ linearly independent over L_0. Of course, these elements are also linearly independent over any subfield of L_0, so since $w(L/F) \ge 2$, we can assume that $r+s+1 \ge w(L/L_0) \ge 2$. In particular, $|L:L_0| \ge 3$. Note that the linear independence over L_0 implies that $\bar{a} \notin L_0\overline{A}$.

Now clearly

$$\begin{aligned} w(E/L_0) &= w(E/L) + w(L/L_0) \\ &\le (r+s+1)t + (r+s+1) = (r+s+1)(t+1). \end{aligned}$$

In particular, since $w(E/F) \ge (r+s+1)^2$, we have

$$\begin{aligned} w(L_0/F) &= w(E/F) - w(E/L_0) \ge (r+s+1)^2 - (r+s+1)(t+1) \\ &= (r+s+1)(r+s-t) \ge r+s+1. \end{aligned}$$

Thus

$$|L_0 : F| \ge 2^{w(L_0/F)} \ge w(L_0/F) + 1 \ge r+s+2.$$

We can now apply Lemma 2.4 with $L \supseteq L_0 \supseteq F$, with elements $\bar{a}, \bar{b} \in L$, and with F-subspaces $\overline{A}, \overline{B} \subseteq L$. Furthermore, we use the linear functional $\lambda \colon L \to L_0$ given by $\lambda = \mathrm{tr}_{L/L_0}$. We conclude that there exists an element $y \in L^\bullet$ such that $\lambda(\bar{a}y) \notin \lambda(\overline{A}y)$, $\lambda(\bar{b}y^{-1}) \notin \lambda(\overline{B}y^{-1})$, and $\dim_F \lambda(\overline{A}y) < \dim_F \overline{A}$.

Finally, observe that $\mu = \lambda \circ \eta = \mathrm{tr}_{L/L_0} \circ \mathrm{tr}_{E/L} = \mathrm{tr}_{E/L_0}$. Furthermore, since $y \in L$ and η is an L-linear functional, we have

$$\mu(axy) = \lambda(\eta(axy)) = \lambda(\eta(ax)y) = \lambda(\bar{a}y).$$

Similarly, $\mu(Axy) = \lambda(\overline{A}y)$, $\mu(bx^{-1}y^{-1}) = \lambda(\bar{b}y^{-1})$, and $\mu(Bx^{-1}y^{-1}) = \lambda(\overline{B}y^{-1})$. Thus, since $\dim_F \lambda(\overline{A}y) < \dim_F \overline{A}$ and $\dim_F \lambda(\overline{B}y^{-1}) \le \dim_F \overline{B}$, we conclude that

$$\dim_F \mu(Axy) + \dim_F \mu(Bx^{-1}y^{-1}) < \dim_F \overline{A} + \dim_F \overline{B} \le r+s-t.$$

The induction step now follows by taking L_0 for the intermediate field and by taking xy for the element of $E^\bullet$. This completes the proof. $\square$

In particular, taking $t = r + s$ in the above result yields

Lemma 3.4. *Let $E \supseteq F$ be finite fields with E/F a square-free extension of width $w(E/F) \geq (r+s+1)^2$, for suitable integers $r, s \geq 0$. Let $a, b \in E$ and let $A, B \subseteq E$, where A and B are F-subspaces of E. Assume that $\dim_F A \leq r$, $\dim_F B \leq s$, $a \notin A$ and $b \notin B$. Then there exists an intermediate field L and an element $x \in E^\bullet$ with*

i. $w(E/L) \leq (r + s + 1)(r + s)$.
ii. $\mathrm{tr}_{E/L}(ax) \neq 0$, *and* $\mathrm{tr}_{E/L}(Ax) = 0$.
iii. $\mathrm{tr}_{E/L}(bx^{-1}) \neq 0$, *and* $\mathrm{tr}_{E/L}(Bx^{-1}) = 0$.

4. Some Generalities

In this section, we let F denote a finite field of characteristic $p > 0$, so that any finite-dimensional F-vector space V is additively an elementary abelian p-group. In addition, we let K be a field of characteristic different from p, and we assume until further notice that K is algebraically closed or at least that it contains a primitive pth root of unity ε. We begin with some basic facts and notation.

Lemma 4.1. *Let V be a finite elementary abelian p-group and let G be a group of automorphisms of V.*

i. The group algebra $K[V]$ is semisimple. Indeed, it is a direct sum of $|V|$ copies of K and every ideal is uniquely an intersection of maximal ideals.
ii. The maximal ideals of $K[V]$ are in one-to-one correspondence with the linear characters $\chi\colon V \to K^\bullet$. To be precise, the ideal $\mathfrak{I}(\chi)$ corresponding to χ is the kernel of the natural algebra extension $\chi\colon K[V] \to K$.
iii. G permutes the linear characters of V by defining $\chi^g(x) = \chi(x^{g^{-1}})$ for all $g \in G$ and $x \in V$. This action corresponds to the permutation action of G on the maximal ideals of $K[V]$.
iv. Every G-stable ideal of $K[V]$ is uniquely an intersection of the maximal G-stable ideals of $K[V]$. The latter are precisely the intersections of G-orbits of maximal ideals of $K[V]$.
v. $\chi^g = \chi$ if and only if $x^{-1}x^{g^{-1}} \in \ker \chi$ for all $x \in V$.

Now suppose $V = V_1 \oplus V_2$ and that $G = F^\bullet$ acts as the torus on V. If $g \in G = F^\bullet$ fixes a character χ of V, then by (v) above, we have

$$\ker \chi \supseteq V_1(g - 1) \oplus V_2(g^{-1} - 1),$$

in additive notation. In particular, if $g \neq 1$, then $\ker \chi \supseteq V$ and we conclude that χ is the trivial (i.e. principal) character of V. In other words, G permutes all the nontrivial characters of V in orbits of full size $|G|$. The following is [5, Lemma 3.2].

Lemma 4.2. *Let F be a finite field with $|F| = q$ and let $G = F^\bullet$ act as the torus on $V = V_1 \oplus V_2$, where both V_1 and V_2 are 1-dimensional.*

i. $K[V_1]$ has precisely two maximal G-stable ideals, namely $\omega(V_1; V_1)$ and one other which we denote by J_1. They satisfy $\omega(V_1; V_1) \cap J_1 = 0$.
ii. $K[V_2]$ has precisely two maximal G-stable ideals, namely $\omega(V_2; V_2)$ and one other which we denote by J_2. They satisfy $\omega(V_2; V_2) \cap J_2 = 0$.

iii. *There are $q+2$ maximal G-stable ideals of $K[V]$. One is $\omega(V;V)$, and two others J_1 and J_2 satisfy $J_1 \cap \omega(V;V) = \omega(V_1;V)$ and $J_2 \cap \omega(V;V) = \omega(V_2;V)$. For convenience, J_1 and J_2 are said to be quasi-augmentation ideals, while the remaining $q-1$ maximal G-stable ideals different from $\omega(V;V)$ are said to be standard.*

We now extend the above to arbitrary finite-dimensional vector spaces. Again parts (i) and (ii) handle the case when only one direct summand occurs, and these results are not really new; indeed, they are contained in the proof of [6, Lemma 2.1]. As with the preceding lemma, this result tells us what to expect and it allows us to introduce some notation, but it does not give us a precise description of the maximal ideals, and hence it will not be used in the proof of our main theorem.

Lemma 4.3. *Let F be a finite field with $|F| = q$ and let $G = F^\bullet$ act as the torus on $V = V_1 \oplus V_2$, where both V_1 and V_2 are nonzero finite-dimensional F-vector spaces.*

i. *Any G-stable ideal of $K[V_1]$ contained in $\omega(V_1;V_1)$ is an intersection of augmentation ideals $\omega(A;V_1)$, where A is an F-subspace of V_1 of codimension at most 1.*

ii. *Any G-stable ideal of $K[V_2]$ contained in $\omega(V_2;V_2)$ is an intersection of augmentation ideals $\omega(B;V_2)$, where B is an F-subspace of V_2 of codimension at most 1.*

iii. *Any G-stable ideal of $K[V]$ contained in $\omega(V;V)$ is an intersection of the following three types of ideals: (1) $J = \omega(A \oplus V_2;V)$, where A is a subspace of V_1 of codimension at most 1; (2) $J = \omega(V_1 \oplus B;V)$, where B is a subspace of V_2 of codimension at most 1; and (3) $J \supseteq \omega(A \oplus B;V)$, where A is a subspace of V_1 of codimension 1, B is a subspace of V_2 of codimension 1, and $J/\omega(A \oplus B;V)$ is a standard ideal of $K[(V_1/A) \oplus (V_2/B)]$.*

Proof. Since every G-stable ideal is an intersection of maximal G-stable ideals, it suffices to determine the latter. If $\Lambda = \Lambda(V)$ denotes the set of nontrivial (i.e. nonprincipal) characters $\lambda\colon V \to K^\bullet$, then $|\Lambda| = |V| - 1$ and we know that G permutes the members of Λ in orbits of full size $|G| = q-1$. Thus, there are precisely $s = (|V|-1)/(q-1)$ such G-orbits on Λ and hence there are precisely s maximal G-stable ideals of $K[V]$ different from $\omega(V;V)$.

(i) Let $|V_1| = q^n$ and, for convenience in this part, write $V = V_1 \oplus 0$. Then we know that there are $s = (q^n-1)/(q-1)$ maximal G-stable ideals of $K[V]$ different from $\omega(V;V)$, and observe that s is precisely the number of subspaces of V of codimension 1. If these subspaces are $A_1, A_2, \ldots, A_s$, then by part (i) of the preceding lemma, $K[V/A_i]$ has precisely two maximal G-stable ideals, namely $\omega(V/A_i;V/A_i)$ and a second one which we denote by $\bar{J}_i$. Furthermore, $\omega(V/A_i;V/A_i) \cap \bar{J}_i = 0$. Lifting these to ideals of $K[V]$, we obtain two maximal G-stable ideals of $K[V]$, namely $\omega(V;V)$ and J_i with $\omega(V;V) \cap J_i = \omega(A_i;V)$.

Since J_i determines A_i, the s ideals $J_1, J_2, \ldots, J_s$ are distinct, and hence these, along with $\omega(V;V)$, are all the maximal G-stable ideals of $K[V]$. Finally, if I is any G-stable ideal of $K[V]$ contained in $\omega(V;V)$, then

$$I = I \cap \omega(V;V) = \bigcap_{i \in T} \omega(V;V) \cap J_i = \bigcap_{i \in T} \omega(A_i;V),$$

where T is a suitable subset of the index set $\{1, 2, \ldots, s\}$. Thus (i) is proved and (ii) follows similarly.

(iii) Let $|V_1| = q^n$ and $|V_2| = q^m$ so that $|V| = |V_1 \oplus V_2| = q^{n+m}$. Then we know that there are precisely $s = (q^{n+m} - 1)/(q - 1)$ maximal G-stable ideals of $K[V]$ different from $\omega(V; V)$. Now there are $(q^n - 1)/(q - 1)$ subspaces of V of the form $A \oplus V_2$, where A has codimension 1 in V_1. As above, each of these subspaces gives rise to a unique maximal G-stable ideal J_1 of $K[V]$ with $\omega(V; V) \cap J_1 = \omega(A \oplus V_2; V)$. Similarly, there are $(q^m - 1)/(q - 1)$ subspaces of V of the form $V_1 \oplus B$, where B has codimension 1 in V_2. Again, each of these gives rise to a unique maximal G-stable ideal J_2 with $\omega(V; V) \cap J_2 = \omega(V_1 \oplus B; V)$. Along with $\omega(V; V)$, we have therefore accounted for

$$1 + \frac{q^n - 1}{q - 1} + \frac{q^m - 1}{q - 1}$$

maximal G-stable ideals of $K[V]$ and there remain

$$t = \frac{q^{n+m} - 1}{q - 1} - \frac{q^n - 1}{q - 1} - \frac{q^m - 1}{q - 1} = \frac{q^n - 1}{q - 1} \cdot \frac{q^m - 1}{q - 1} \cdot (q - 1)$$

ideals that must be described.

Now there are precisely

$$\frac{q^n - 1}{q - 1} \cdot \frac{q^m - 1}{q - 1}$$

subspaces of V of the form $A \oplus B$, where A has codimension 1 in V_1 and B has codimension 1 in V_2. By part (iii) of the preceding lemma, $K[(V_1/A) \oplus (V_2/B)]$ has $q - 1$ standard maximal G-stable ideals, and lifting these to $K[V] = K[V_1 \oplus V_2]$, we obtain potentially

$$t = \frac{q^n - 1}{q - 1} \cdot \frac{q^m - 1}{q - 1} \cdot (q - 1)$$

new maximal G-stable ideals of $K[V]$. For this count to be exact, we must check that none of these ideals appears among the J_1 and J_2 ideals mentioned above, and that each of these t possible ideals is distinct.

To this end, we use the fact that if I is an ideal of $K[V]$ with $I \supseteq \omega(X; V)$ and $I \supseteq \omega(Y; V)$, then $I \supseteq \omega(X; V) + \omega(Y; V) = \omega(Z; V)$ where $Z = X + Y$. For example, suppose I is one of the standard ideals above with $I \supseteq \omega(A \oplus B; V)$ and suppose that I is also one of the J_1 or J_2 ideals, say $I \supseteq \omega(A' \oplus V_2; V)$. Since $A' \oplus V_2$ is a maximal subspace of V and since I does not contain $\omega(V; V)$, it follows that $A' \oplus V_2 \supseteq A \oplus B$ and hence $A' = A$. But then I corresponds to both a quasi-augmentation and a standard ideal of $K[(V_1/A) \oplus (V_2/B)]$, and this is a contradiction. Thus we see that all the standard ideals above are new. It remains to show that they are all distinct. For this, suppose I is an standard ideal with $I \supseteq \omega(A \oplus B; V)$ and also $I \supseteq \omega(A' \oplus B'; V)$. Then $I \supseteq \omega(\overline{A} \oplus \overline{B}; V)$ where $\overline{A} = A + A'$ and $\overline{B} = B + B'$. But, as we have seen, we cannot have $\overline{A} = V_1$ or $\overline{B} = V_2$. Thus, $A = A'$ and $B = B'$, so we conclude that there is no overlap in the count for t. In particular, we have now accounted for all the maximal G-stable ideals of $K[V]$, and the result clearly follows. □

To reiterate, there are three types of maximal G-stable ideals of $K[V]$ in the above context. First, we have $\omega(V; V)$. Next, we have the *quasi-augmentation ideals* J. These contain a unique augmentation ideal $\omega(W; V)$, where W is a G-stable subspace of V of codimension 1 and $J \cap \omega(V; V) = \omega(W; V)$. In this case, we say that $\omega(W; V)$ is the augmentation ideal *covered* by J. Finally, we have the *standard* ideals S. These contain a unique augmentation ideal $\omega(W; V)$, where W

is a G-stable subspace of V of codimension 2, but they do not contain $\omega(W';V)$ with W' of codimension 1 in V. Again, in this case, we say that $\omega(W;V)$ is the augmentation ideal *covered* by S.

Another consequence of Lemma 4.1 is the almost obvious

Lemma 4.4. *Let $V \supseteq W$ be finite elementary abelian p-groups and let I be an ideal of $K[V]$.*

i. Let $\chi\colon V \to K^\bullet$ be a character of V corresponding to a maximal ideal that contains I. Then the ideal corresponding to the restricted character $\chi_W\colon W \to K^\bullet$ contains $I \cap K[W]$.

ii. Conversely, let $\theta\colon W \to K^\bullet$ be a character of W whose corresponding ideal contains $I \cap K[W]$. Then there exists a character $\chi\colon V \to K^\bullet$ with $\chi_W = \theta$ and with the ideal corresponding to χ containing I.

Proof. Again, we use $\mathfrak{I}(\chi)$ to denote the ideal of $K[V]$ corresponding to χ. Then certainly $\mathfrak{I}(\chi) \cap K[W] = \mathfrak{I}(\chi_W)$ is the ideal of $K[W]$ corresponding to the restriction χ_W. Now let I be an ideal of $K[V]$ and let S denote the set of characters χ of V with $\mathfrak{I}(\chi) \supseteq I$. Then $I = \bigcap_{\chi\in S} \mathfrak{I}(\chi)$ so

$$I \cap K[W] = \bigcap_{\chi\in S} \mathfrak{I}(\chi) \cap K[W] = \bigcap_{\chi\in S} \mathfrak{I}(\chi_W),$$

and (i) follows. Conversely, if $\mathfrak{I}(\theta)$ contains $I \cap K[W]$, then the uniqueness aspect of Lemma 4.1(i) implies that $\theta = \chi_W$ for some $\chi \in S$. $\square$

5. G-Stable Ideals

Let $\mathbf{F}$ denote the algebraic closure of the prime field $\mathrm{GF}(p)$ or perhaps just a wide subfield of the algebraic closure. In this section, we study the torus action of $\mathbf{F}^\bullet$ on $\mathbf{F}^n$, the vector space of n-tuples. This space has a natural basis $\kappa_1, \kappa_2, \ldots, \kappa_n$ and, if F is a subfield of $\mathbf{F}$, then $\kappa_1 F + \kappa_2 F + \cdots + \kappa_n F = F^n$ is contained naturally in $\mathbf{F}^n$. Now suppose we change the F-basis of F^n to $\nu_1, \nu_2, \ldots, \nu_n$. Then we obtain a new basis for $\mathbf{F}^n$ and indeed, if E is an intermediate field with $\mathbf{F} \supseteq E \supseteq F$, then E^n is the same space whether viewed in the original basis or this new one. On the other hand, if $\mathbf{F} \supseteq E$ but $E \not\supseteq F$, then the E-space E^n may have changed dramatically in this process, and so these fields must be avoided. In other words, if we make a basis change in F^n, then we can only safely consider the subfields E of $\mathbf{F}$ that contain F, but no others.

Let F be a finite subfield of $\mathbf{F}$. Then Lemma 4.3 gives us a quick count of the maximal G-stable ideals of $K[V]$, but it does not really give us a good description of them. So we take a closer look at the action of $G = F^\bullet$ on $V = F^n$ and on $K[V]$. As we see below, an appropriate basis change allows us to better understand the maximal G-stable ideals.

Example 5.1. Let F be a finite field and let $V = V_1 \oplus V_2 = F^{r+1} \oplus F^{s+1} = F^n$, where $\dim_F V_1 = r+1 \geq 1$, $\dim_F V_2 = s+1 \geq 1$ and $\dim_F V = r+s+2 = n$. For convenience, we let $\mathbf{a}$ and $\mathbf{x}$ denote the $(r+1)$-tuples $(a_0, a_1, \ldots, a_r)$ and $(x_0, x_1, \ldots, x_r)$ respectively, while $\mathbf{b}$ and $\mathbf{y}$ denote the $(s+1)$-tuples $(b_0, b_1, \ldots, b_s)$ and $(y_0, y_1, \ldots, y_s)$ respectively. Then the torus action of $F^\bullet$ on F^n is given by $(\mathbf{x}, \mathbf{y})^g = (\mathbf{x}g^{-1}, \mathbf{y}g)$ for all $\mathbf{x} \in F^{r+1}$, $\mathbf{y} \in F^{s+1}$ and $g \in F^\bullet$. Of course, $F^\bullet$ also acts on the group algebra $K[F^n]$ and our goal here is to describe the maximal $F^\bullet$-stable ideals of $K[F^n]$. At this point, we assume that $\operatorname{char} F = p > 0$, $\operatorname{char} K \neq p$ and that K contains ε, a primitive pth root of unity.

Let $\mathrm{GF}(p)$ denote the prime subfield of F and let $\mu\colon F \to \mathrm{GF}(p)$ be a nonzero $GF(p)$-linear functional. Then, by Lemma 2.1, all linear functionals from F to $\mathrm{GF}(p)$ are of the form $\mu_a\colon F \to \mathrm{GF}(p)$, where $a \in F$ and $\mu_a(x) = \mu(ax)$. Hence all characters $\chi\colon F \to K^\bullet$ are given by $\chi_a(x) = \varepsilon^{\mu_a(x)} = \varepsilon^{\mu(ax)}$. Furthermore, since the characters from F^n to $K^\bullet$ are necessarily products, they are all of the form

$$\begin{aligned}\chi_{\mathbf{a},\mathbf{b}}(\mathbf{x},\mathbf{y}) &= \varepsilon^{\sum_i \mu(a_i x_i) + \sum_j \mu(b_j y_j)}\\ &= \varepsilon^{\mu(\sum_i a_i x_i + \sum_j b_j y_j)} = \varepsilon^{\mu(\mathbf{a}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{y})},\end{aligned}$$

where we use $\cdot$ to denote the usual dot product. These characters in turn extend naturally to the K-algebra homomorphisms $\chi_{\mathbf{a},\mathbf{b}}\colon K[F^n] \to K$ and their kernels $I_{\mathbf{a},\mathbf{b}} = \ker \chi_{\mathbf{a},\mathbf{b}} = \mathfrak{I}(\chi_{\mathbf{a},\mathbf{b}})$ are precisely the set of maximal ideals of $K[F^n]$.

Now $F^\bullet$ permutes these characters by

$$\begin{aligned}\chi^g_{\mathbf{a},\mathbf{b}}(\mathbf{x},\mathbf{y}) &= \chi_{\mathbf{a},\mathbf{b}}((\mathbf{x},\mathbf{y})^{g^{-1}}) = \chi_{\mathbf{a},\mathbf{b}}(g\mathbf{x}, g^{-1}\mathbf{y})\\ &= \varepsilon^{\mu(\mathbf{a}\cdot g\mathbf{x}+\mathbf{b}\cdot g^{-1}\mathbf{y})} = \chi_{\mathbf{a}g,\mathbf{b}g^{-1}}(\mathbf{x},\mathbf{y})\end{aligned}$$

for all $g \in F^\bullet$, and hence we have $I^g_{\mathbf{a},\mathbf{b}} = I_{\mathbf{a}g,\mathbf{b}g^{-1}}$. With this, it is easy to describe the $F^\bullet$-orbits of these ideals, but we will state the precise result only for certain special cases.

If $\mathbf{a} = \mathbf{b} = \mathbf{0}$, then $\chi_{\mathbf{0},\mathbf{0}}$ is certainly the trivial character of $V = F^n$ and hence $I_{\mathbf{0},\mathbf{0}}$ is the augmentation ideal $\omega(V;V)$. Next, suppose just one of $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, say $\mathbf{b} = \mathbf{0}$. Notice that the map $\mathbf{x} \mapsto \mathbf{x}\cdot\mathbf{a}$ is a nonzero linear functional on V_1, and let W_1 be the kernel of this map. Then W_1 is a subspace of V_1 of codimension 1, and $W_1 \oplus V_2$ is a subspace of $V = V_1 \oplus V_2$ of codimension 1. Furthermore, if we change the basis of V_1 so that it ends with a basis for W_1, then the character $\chi_{\mathbf{a},\mathbf{0}}$ now looks like

$$\chi_{\mathbf{a},\mathbf{0}}(\mathbf{x},\mathbf{y}) = \varepsilon^{\mu(a_0 x_0)}$$

with $0 \neq a_0 \in F$. It is clear that the $F^\bullet$-orbit of $\chi_{\mathbf{a},\mathbf{0}}$ consists of the characters of the form $x \mapsto \varepsilon^{\mu(a x_0)}$ for all $0 \neq a \in F$. Certainly, the intersection of the maximal ideals corresponding to the members of this orbit is a quasi-augmentation ideal that covers $\omega(W_1 \oplus V_2; V)$. Of course, a similar structure holds if $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$.

Finally, the generic or standard case occurs when $\mathbf{a}$ and $\mathbf{b}$ are both not zero. Here we let W_1 be the kernel of the functional $V_1 \to F$ given by $\mathbf{x} \mapsto \mathbf{a}\cdot\mathbf{x}$, and we let W_2 be the kernel of the functional $V_2 \to F$ given by $\mathbf{y} \mapsto \mathbf{b}\cdot\mathbf{y}$. Then W_1 and W_2 are subspaces of codimension 1 in V_1 and V_2, respectively, so $W_1 \oplus W_2$ has codimension 2 in $V_1 \oplus V_2 = V$. Furthermore, if we change the bases of V_1 and V_2 so that they end with bases for W_1 and W_2, respectively, then the character $\chi_{\mathbf{a},\mathbf{b}}$ now looks like

$$\chi_{\mathbf{a},\mathbf{b}}(\mathbf{x},\mathbf{y}) = \varepsilon^{\mu(a_0 x_0 + b_0 y_0)}$$

with $0 \neq a_0, b_0 \in F$. It is clear that the $F^\bullet$-orbit of $\chi_{\mathbf{a},\mathbf{b}}$ consists of the characters of the form $x \mapsto \varepsilon^{\mu(a_0 g x_0 + b_0 g^{-1} y_0)}$ for all $g \in F^\bullet$. Certainly, the intersection of the maximal ideals corresponding to the members of this orbit is a standard ideal that covers $\omega(W_1 \oplus W_2; V)$. In addition, it is clear that $\omega(W_1 \oplus W_2; V)$ is the intersection of the $|F|^2$ maximal ideals of $K[V]$ corresponding to the characters $(\mathbf{x},\mathbf{y}) \mapsto \varepsilon^{\mu(a x_0 + b y_0)}$ for all $a, b \in F$.

In view of Lemma 4.4, it is appropriate to see how the characters described above restrict to subfields. We do this in the special case where the functionals are built from Galois traces, but the observation is true more generally.

Lemma 5.2. *Let $F \subseteq L \subseteq E$ be finite subfields of $\mathbf{F}$ and fix a nonzero linear functional $\mu\colon F \to \mathrm{GF}(p)$. Then we have nonzero linear functionals $\lambda\colon E \to \mathrm{GF}(p)$ and $\eta\colon L \to \mathrm{GF}(p)$ defined by $\lambda = \mu\circ\mathrm{tr}_{E/F}$ and $\eta = \mu\circ\mathrm{tr}_{L/F}$. If $\chi_{\mathbf{a},\mathbf{b}}$ is a character of $E^n = E^{r+1} \oplus E^{s+1}$ given by*

$$\chi_{\mathbf{a},\mathbf{b}}(\mathbf{x}, \mathbf{y}) = \varepsilon^{\lambda(\mathbf{a}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{y})},$$

then the restriction $\widetilde{\chi}$ of $\chi_{\mathbf{a},\mathbf{b}}$ to L^n is given by

$$\widetilde{\chi}(\mathbf{x}, \mathbf{y}) = \varepsilon^{\eta(\tilde{\mathbf{a}}\cdot\mathbf{x}+\tilde{\mathbf{b}}\cdot\mathbf{y})} = \widetilde{\chi}_{\tilde{\mathbf{a}},\tilde{\mathbf{b}}}(\mathbf{x}, \mathbf{y}),$$

where $\tilde{\mathbf{a}} = \mathrm{tr}_{E/L}(\mathbf{a})$ and $\tilde{\mathbf{b}} = \mathrm{tr}_{E/L}(\mathbf{b})$. Furthermore, if p does not divide $|E : F|$, then this applies equally well when the trace is replaced by the normalized trace.

Proof. Since $\mathrm{tr}_{E/F} = \mathrm{tr}_{L/F}\circ\mathrm{tr}_{E/L}$, it follows that $\lambda = \eta\circ\mathrm{tr}_{E/L}$. In particular, since $\mathrm{tr}_{E/L}$ is an L-linear functional, we see that if $\mathbf{x} \in L^{r+1}$ and $\mathbf{y} \in L^{s+1}$, then

$$\begin{aligned}\lambda(\mathbf{a}\cdot\mathbf{x} + \mathbf{b}\cdot\mathbf{y}) &= \eta(\mathrm{tr}_{E/L}(\mathbf{a}\cdot\mathbf{x} + \mathbf{b}\cdot\mathbf{y})) \\ &= \eta(\mathrm{tr}_{E/L}(\mathbf{a})\cdot\mathbf{x} + \mathrm{tr}_{E/L}(\mathbf{b})\cdot\mathbf{y}) = \eta(\tilde{\mathbf{a}}\cdot\mathbf{x} + \tilde{\mathbf{b}}\cdot\mathbf{y}).\end{aligned}$$

Here of course $\mathrm{tr}_{E/L}$ acts on $\mathbf{a}$ and $\mathbf{b}$ by acting on each entry. □

Our goal now is to obtain an appropriate analog of [5, Lemma 3.4]. As will be apparent, the proof first reduces the general problem to the 2-dimensional case where [5, Lemma 3.4] should apply. Unfortunately, the latter lemma is not stated in a way that can be directly quoted here. So, we are forced to skim through its argument, to see that it yields what we want.

Lemma 5.3. *Let $\mathbf{F}^\bullet$ act as the torus on $\mathbf{F}^{r+1}\oplus\mathbf{F}^{s+1} = \mathbf{F}^n$ and let I be an $\mathbf{F}^\bullet$-stable ideal of $K[\mathbf{F}^n]$. Suppose F is a finite subfield of $\mathbf{F}$ and let S be a standard maximal $F^\bullet$-stable ideal of $K[F^n]$ that contains $I\cap K[F^n]$. If S covers the augmentation ideal $\omega(W_1 \oplus W_2; V)$, where $V = F^n = F^{r+1} \oplus F^{s+1}$, then $\omega(W_1 \oplus W_2; V) \supseteq I \cap K[F^n]$.*

Proof. Since $\mathbf{F}$ is a wide subfield of the algebraically closure of $\mathrm{GF}(p)$, we can find a finite extension $\overline{E}$ of F, with $\overline{E} \subseteq \mathbf{F}$, and such that $\overline{E}/F$ is square-free of width

$$w(\overline{E}/F) \geq n^2 + 2|F|^2.$$

Furthermore, we can assume that p does not divide $|\overline{E} : F|$. Fix $\mu\colon F \to \mathrm{GF}(p)$, a $\mathrm{GF}(p)$-linear functional with $\mu(1) = 1$. In addition, for each intermediate field L with $\overline{E} \supseteq L \supseteq F$, let $\lambda_L\colon L \to \mathrm{GF}(p)$ be the composite $\lambda_L = \mu\circ\overline{\mathrm{tr}}_{L/F}$. Using these functionals, we can describe each group algebra $K[L^n]$ as in Example 5.1.

Indeed, in view of that example and by way of a change of basis in F^n, we can assume that one character in the $F^\bullet$-orbit that defines S is of the form

$$\chi_{\mathbf{a}',\mathbf{b}'}(\mathbf{x}, \mathbf{y}) = \varepsilon^{\mu(a'_0x_0+b'_0y_0)}$$

with $0 \neq a'_0, b'_0 \in F$. Our goal is to show that for all $a', b' \in F$, the ideal associated with the character

$$(\mathbf{x}, \mathbf{y}) \mapsto \varepsilon^{\mu(a'x_0+b'y_0)}$$

contains $I \cap K[F^n]$. If we succeed, then the result will follow since Example 5.1 implies that all these associated ideals intersect to $\omega(W_1 \oplus W_2; V)$.

By Lemma 4.4, there exists a character $\overline{\chi}$ of $\overline{E}^n$ so that $\mathfrak{I}(\overline{\chi}) \supseteq I \cap K[\overline{E}^n]$ and $\overline{\chi}$ restricted to F^n is $\chi_{\mathbf{a}',\mathbf{b}'}$. Say

$$\overline{\chi}(\mathbf{x},\mathbf{y}) = \overline{\chi}_{\mathbf{a},\mathbf{b}}(\mathbf{x},\mathbf{y}) = \varepsilon^{\lambda_{\bar{E}}(\mathbf{a}\cdot\mathbf{x}+\mathbf{b}\cdot\mathbf{y})},$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_r) \in \overline{E}^{r+1}$ and $\mathbf{b} = (b_0, b_1, \ldots, b_s) \in \overline{E}^{s+1}$. Since $\overline{\chi}$ restricts to $\chi_{\mathbf{a}',\mathbf{b}'}$, Lemma 5.2 implies that $\overline{tr}_{\bar{E}/F}(a_0) = a_0' \neq 0$, $\overline{tr}_{\bar{E}/F}(a_i) = 0$ for $1 \le i \le r$, $\overline{tr}_{\bar{E}/F}(b_0) = b_0' \neq 0$, and $\overline{tr}_{\bar{E}/F}(b_j) = 0$ for $1 \le j \le s$.

Now let $A = Fa_1 + Fa_2 + \cdots + Fa_r$ and $B = Fb_1 + Fb_2 + \cdots + Fb_r$ so that A and B are F-subspaces of $\overline{E}$ with $\mathrm{tr}_{\bar{E}/F}(A) = \mathrm{tr}_{\bar{E}/F}(B) = 0$, $\dim_F A \le r$ and $\dim_F B \le s$. Clearly, $a_0 \notin A$, $b_0 \notin B$ and $w(\overline{E}/F) \ge n^2 > (r+s+1)^2$. Lemma 3.4 now implies that there exists an intermediate field E, with $w(\overline{E}/E) \le (r+s+1)(r+s)$, and an element $g \in \overline{E}^\bullet$ such that $\mathrm{tr}_{\bar{E}/E}(a_0 g) = a \neq 0$, $\mathrm{tr}_{\bar{E}/E}(b_0 g^{-1}) = b \neq 0$, and $\mathrm{tr}_{\bar{E}/E}(Ag) = \mathrm{tr}_{\bar{E}/E}(Bg^{-1}) = 0$. In particular, by Lemma 5.2, the restriction of $\overline{\chi}^g$ to E^n satisfies

$$\widetilde{\chi}(\mathbf{x},\mathbf{y}) = \varepsilon^{\lambda_E(ax_0+by_0)}.$$

Note that $I \cap K[\overline{E}^n]$ is $\overline{E}^\bullet$-stable, so the ideal corresponding to $\overline{\chi}^g$ also contains $I \cap K[\overline{E}^n]$. Consequently, the ideal of $K[E^n]$ corresponding to the character $\widetilde{\chi}$ contains $I \cap K[E^n]$. On the other hand, there is no reason to believe that the restriction of $\widetilde{\chi}$ to F^n is in the $F^\bullet$-orbit of $\chi_{\mathbf{a}',\mathbf{b}'}$.

At this point, we have essentially reduced the problem to the 2-dimensional case, since the characters depend only upon x_0 and y_0. We now follow the argument of [5, Lemma 3.4]. Since $w(\overline{E}/F) \ge n^2 + 2|F|^2$ and $w(\overline{E}/E) \le (r+s+1)(r+s) = (n-1)(n-2)$, it is clear that $w(E/F) \ge 2 + 2|F|^2$. In particular, we can choose an intermediate field L with $E \supseteq L \supseteq F$, $w(E/L) = 2$ and $w(L/F) \ge 2|F|^2$. Thus $|E : L| \ge 3$ and $|L : F| \ge 2^{w(L/F)} \ge w(L/F) + 1 \ge 2|F|^2 + 1$.

Let us write $\widetilde{\chi}(\mathbf{x},\mathbf{y}) = \varepsilon^{\eta_E(ax_0+bx_0)}$ as $\widetilde{\chi}_{a,b}(\mathbf{x},\mathbf{y})$. Then $\mathfrak{I}(\widetilde{\chi}_{a,b}) \supseteq I \cap K[E^n]$ and the latter ideal is $E^\bullet$-stable. Thus, for all $z \in E^\bullet$, we see that the ideal associated with $\widetilde{\chi}^z_{a,b} = \widetilde{\chi}_{az,bz^{-1}}$ also contains $I \cap K[E^n]$. In particular, if we replace z by bz^{-1}, then we see that the $E^\bullet$-orbit of $\widetilde{\chi}_{a,b}$ consists of all $\widetilde{\chi}_{sz^{-1},z}$, where $s = ab \neq 0$. Of course, all the ideals associated with these characters contain $I \cap K[E^n]$. Now define

$$H = \{\overline{tr}_{E/L}(sz^{-1}) \mid z \in E,\ \overline{tr}_{E/L}(z) = 1\}.$$

Since $s \neq 0$ and $|E : L| \ge 3$, [5, Lemma 2.2] implies that $0 \in H$ and $|H| \ge \sqrt{|L|/2}$. Furthermore, by Lemma 5.2, the restriction of any $\widetilde{\chi}_{sz^{-1},z}$ to L^n is of the form

$$\chi'_{u,v}(\mathbf{x},\mathbf{y}) = \varepsilon^{\eta_L(ux_0+vy_0)},$$

where $u = \overline{tr}_{E/L}(sz^{-1})$ and $v = \overline{tr}_{E/L}(z)$. In particular, by limiting our considerations to those z with $\overline{tr}_{E/L}(z) = 1$, we obtain restricted characters of the form $\chi'_{h,1}(\mathbf{x},\mathbf{y})$ for all $h \in H$. Of course, the ideal associated to each $\chi'_{h,1}$ contains $I \cap K[L^n]$, and the $\chi'_{h,1}$ are all in different $L^\bullet$-orbits. Thus we obtain a reasonably large number of $L^\bullet$-orbits in this restriction, but not enough to finish the problem.

So, we require a second pass, this time from L to F. Write $|F| = q$, $|L : F| = \ell$ and $m = q^2 = |F|^2$. Then $\ell = |L : F| \ge 2|F|^2 + 1 = 2m + 1$, so $|L|/2 \ge q^{2m+1}/2 \ge q^{2m}$, and hence $|H| \ge \sqrt{|L|/2} \ge q^m$. It follows that the F-linear span of the elements of H has F-dimension at least m, and hence we can choose

$h_1, h_2, \ldots, h_m \in H$ so that $M = \{h_1, h_2, \ldots, h_m\}$ is an F-linearly independent subset of L of size m. Since $m = q^2$, it now follows from [5, Lemma 2.3] that

$$\mathrm{Non}(M) = \{\tau \colon L \to F \mid \tau(M) < F\}$$

has size strictly less than $q^{\ell-1}$. Here, of course, each such τ is an F-linear functional, and "Non" is an abbreviation for "not onto".

On the other hand, note that $C = \{c \in L \mid \overline{tr}_{L/F}(c) = 1\}$ is a coset of the kernel of $\overline{tr}_{L/F}$ and hence has size $q^{\ell-1}$. Thus, since $0 \notin C$, this set gives rise to precisely $q^{\ell-1}$ distinct F-linear functionals $\tau \colon L \to F$ by taking $\tau(x) = (\overline{tr}_{L/F})_{c^{-1}}(x) = \overline{tr}_{L/F}(c^{-1}x)$ for all $c \in C$. But $|\mathrm{Non}(M)| < q^{\ell-1}$, so there exists $d \in C$ with $\overline{tr}_{L/F}(d^{-1}M) = F$ and hence with $\overline{tr}_{L/F}(d^{-1}H) = F$. Of course, $d \in C$ says that $\overline{tr}_{L/F}(d) = 1$.

Now, we know that $\tilde{I} = I \cap K[L^n]$ is an $L^\bullet$-stable ideal of $K[L^n]$ that is contained in the maximal $L^\bullet$-stable ideals corresponding to at least the characters $\tilde{\chi}_{h,1}$ with $h \in H$. Thus $\tilde{I}$ is contained in the kernel of the algebra homomorphisms corresponding to the characters

$$\tilde{\chi}_{hd^{-1},d}(\mathbf{x}, \mathbf{y}) = \varepsilon^{\lambda_L(hd^{-1}x_0 + dy_0)}$$

for all $h \in H$. Again, since $\overline{tr}_{L/F}(d) = 1$, Lemma 5.2 implies that the restrictions of these characters to F^n are all of the form

$$\chi(\mathbf{x}, \mathbf{y}) = \varepsilon^{\mu(tx_0 + y_0)} = \chi_{t,1}(\mathbf{x}, \mathbf{y}),$$

where $t = \overline{tr}_{L/F}(hd^{-1})$. Indeed, since $\overline{tr}_{L/F}(Hd^{-1}) = F$, and $I \cap K[F^n]$ is $F^\bullet$-stable, we conclude that $I \cap K[F^n]$ is contained in the ideals associated to $\chi_{t,1}^f = \chi_{tf,f^{-1}}$ for all $t \in F$ and $f \in F^\bullet$. In particular, the ideals associated with all characters of the form $(\mathbf{x}, \mathbf{y}) \mapsto \varepsilon^{\mu(a'x_0 + b'y_0)}$ with $a' \in F$ and $0 \neq b' \in F$ contain $I \cap K[F^n]$. But this situation is really right-left symmetric, so all ideals associated to characters of the form $(\mathbf{x}, \mathbf{y}) \mapsto \varepsilon^{\mu(a'x_0 + b'y_0)}$ with $0 \neq a' \in F$ and $b' \in F$ also contain $I \cap K[F^n]$.

This leaves only the trivial character $(\mathbf{x}, \mathbf{y}) \mapsto 1$. For this, note that $0 \in H$, so that $I \cap K[L^n]$ is contained in the ideal associated with the character $\chi'_{0,1}$ and hence with any character of the form $\chi'_{0,j}$ with $0 \neq j \in L$. But then we can surely take j in the kernel of $\overline{tr}_{L/F}$ and hence this character restricts to the trivial character of F^n. With this, we conclude that the ideals associated with all characters of the form $(\mathbf{x}, \mathbf{y}) \mapsto \varepsilon^{\mu(a'x_0 + b'y_0)}$ with $a', b' \in F$ contain $I \cap K[F^n]$ and hence $\omega(W_1 \oplus W_2; V) \supseteq I \cap K[F^n]$, as required. $\square$

We could certainly prove an analogous result for the quasi-augmentation ideals, but it is not difficult to avoid that additional unpleasantness. Indeed, we can now prove the following reformulation of the main theorem, using slightly different notation.

Proposition 5.4. *Let $\mathbf{F}$ be an infinite locally-finite field, let V_1 and V_2 be two finite-dimensional $\mathbf{F}$-vector spaces, and set $V = V_1 \oplus V_2$. If $G = \mathbf{F}^\bullet$, then we can let G act as the torus on V, and hence G acts on the group algebra $K[V]$. Suppose, in addition, that $\operatorname{char} K \neq \operatorname{char} \mathbf{F}$ and that $\mathbf{F}$ is wide. Then every G-stable ideal of $K[V]$ is uniquely a finite irredundant intersection of augmentation ideals $\omega(W_1 \oplus W_2; V)$ with W_1 an $\mathbf{F}$-subspace of V_1 and W_2 an $\mathbf{F}$-subspace of V_2. As a consequence, the set of G-stable ideals of $K[V]$ is Noetherian.*

Proof. Let $\operatorname{char}\mathbf{F} = p > 0$, so that $\mathbf{F}$ is a wide subfield of the algebraic closure of $\mathrm{GF}(p)$, and suppose for now that the field K contains a primitive pth root of 1. If either V_1 or V_2 is 0, then $\mathbf{F}^\bullet$ acts on V as scalar multiplication, so the result follows from [6, Theorem A]. Thus, we can assume that $\dim_{\mathbf{F}} V_1 = r+1 \geq 1$ and $\dim_{\mathbf{F}} V_2 = s+1 \geq 1$, so $V = \mathbf{F}^n$ with $n = r+s+2$. Of course, $\mathbf{F}$ is an ascending union of a family $\mathcal{F}$ of finite subfields, and using the natural basis for $\mathbf{F}^n$, it is clear that the set $\{(F^n, F^\bullet) \mid F \in \mathcal{F}\}$, is a local system for (V,G) in the notation of [6, Section 1].

Let I be an $\mathbf{F}^\bullet$-stable ideal of $K[V]$ and assume that I is contained in the augmentation ideal $\omega(V;V)$. Then, for any $F \in \mathcal{F}$, $I \cap K[F^n]$ is an $F^\bullet$-stable ideal of $K[F^n]$ contained in $\omega(F^n;F^n)$. Thus $I \cap K[F^n]$ is a finite intersection $\bigcap_{i=0}^k M_i$ of maximal $F^\bullet$-stable ideals of $K[F^n]$ with $M_0 = \omega(F^n;F^n)$. For $i \neq 0$, let $\omega(U_i;F^n)$ be the augmentation ideal covered by M_i, so that U_i is a subspace of F^n of codimension 1 or 2. Since $M_i \supseteq \omega(U_i;F^n)$, we have

$$I \cap K[F^n] \supseteq \omega(F^n;F^n) \cap \bigcap_{i=1}^k \omega(U_i;F^n).$$

Now if M_i is a quasi-augmentation ideal, then $\omega(U_i;F^n) = M_i \cap \omega(F^n;F^n) \supseteq I \cap K[F^n]$. On the other hand, if M_i is standard, then the previous lemma asserts that $\omega(U_i;F^n) \supseteq I$. We conclude that

$$\omega(F^n;F^n) \cap \bigcap_{i=1}^k \omega(U_i;F^n) \supseteq I \cap K[F^n]$$

and hence equality occurs.

[6, Lemma 1.2] now implies that I is an intersection (possibly infinite) of augmentation ideals corresponding to G-stable subgroups of V, and by Lemma 1.2, these subgroups are all of the form $W_1 \oplus W_2$, with W_1 and $\mathbf{F}$-subspace of V_1 and with W_2 and $\mathbf{F}$-subspace of V_2. Furthermore, since $\dim_{\mathbf{F}} V < \infty$, it follows that V has finite composition length as a G-module. Thus we can conclude from [6, Lemma 1.6] that I is a finite intersection of augmentation ideals determined by G-stable subgroups of V.

Next, we show that any G-stable ideal of $K[V]$ is contained in $\omega(V;V)$ and hence has the above structure. To this end, observe from Lemma 1.2 that all G-sections of V are infinite. Hence by [6, Lemma 1.3], if A is any G-stable subgroup of V, then $\omega(A;V)$ is a G-prime ideal of $K[V]$. In particular, $K[V]$ itself is G-prime. Now let I be any G-stable ideal of $K[V]$ and set $J = I{\cdot}\omega(V;V)$, so that J is a G-stable ideal of $K[V]$ contained in $\omega(V;V)$. If $J = \omega(V;V)$, then $I \supseteq J$ implies that $I = \omega(V;V)$. Otherwise, by the result of the previous paragraph, $J \subseteq \omega(A;V)$ for some G-stable subgroup A properly smaller than V. But $\omega(A;V)$ is a G-prime ideal and $I{\cdot}\omega(V;V) \subseteq \omega(A;V)$, so we conclude that $I \subseteq \omega(A;V) \subseteq \omega(V;V)$, as required.

It remains to consider arbitrary fields K with $\operatorname{char} K \neq p$. In this situation, we let $\overline{K}$ be an extension of K that contains a primitive pth root of unity. If I is a G-stable ideal of $K[V]$, then $\overline{K}{\cdot}I$ is a G-stable ideal of $\overline{K}[V]$, and the freeness of $\overline{K}$ over K easily implies that $(\overline{K}{\cdot}I) \cap K[V] = I$. Since $\overline{K}{\cdot}I$ is a finite intersection of augmentation ideals $\omega_{\overline{K}}(A;V)$ and since $\omega_{\overline{K}}(A;V) \cap K[V] = \omega(A;V)$, we see that I is also a finite intersection of augmentation ideals. This intersection can of course be made irredundant, and then uniqueness follows from [6, Lemma 1.4]. Finally, the set of these ideals is Noetherian by [6, Lemma 1.8]. □

6. Final Comments

As we indicated earlier, it is not clear whether the wideness hypothesis is really required for Theorem 1.1. There is a trick in [2, Lemma 3.7] which shows, for the scalar action of $F^{\bullet}$ on V, that the validity of results like [2, Theorem B] are inherited from large fields to arbitrary infinite subfields. Unfortunately, however, one can "prove" that this argument will not apply to the torus action of $F^{\bullet}$. Still, similar tricks of this nature might yet exist.

Finally, the author would like to thank Prof. J. M. Osterburg for interesting conversations on this work.

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