THE SEMIPRIMITIVITY OF GROUP ALGEBRAS

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Abstract. In this paper we briefly discuss some recent results on the semiprimitivity problem for group algebras. For the most part, we stress those topics which do not appear in the fairly complete survey [P17]. In particular, we consider the controller of an ideal, examples associated with the controller subgroup of the Jacobson radical, and a new, but rather elementary, observation on Kaplansky's problem.

§1. The Controller of an Ideal

Let $K[G]$ denote the group algebra of the multiplicative group G over the field K. Our goal is to determine when $K[G]$ is semiprimitive and, more generally, to describe its Jacobson radical $\mathcal{J}K[G]$. Obviously, we would expect the latter description to be sufficiently precise to allows us to immediately decide whether or not the radical is zero. In particular, we need some sort of mechanism for describing an arbitrary two-sided ideal of a group algebra.

We first recall some standard notation. If $\alpha = \sum_{x\in G} a_x x \in K[G]$, then the *support* of α is given by the finite set $\operatorname{supp}\alpha = \{\, x \in G \mid a_x \neq 0\,\}$. Furthermore, if H is any subgroup of G, then there is a natural $K[H]$-bimodule projection map $\pi_H: K[G] \to K[H] \subseteq K[G]$ given by $\pi_H: \sum_{x\in G} a_x x \mapsto \sum_{x\in H} a_x x$. Thus π_H is the K-linear extension of the map $G \to H \cup \{\,0\,\}$ which is the identity on H and zero on $G \setminus H$. In particular, $\pi_H(\alpha)$ is a "truncation" of α, and if A and B are subgroups of G, then $\pi_A(\pi_B(\alpha)) = \pi_{A\cap B}(\alpha)$.

Now let $I \triangleleft K[G]$ and let $H \triangleleft G$. Then H *controls* I if $I = (I \cap K[H])\cdot K[G]$, that is, if $K[H]$ contains generators for I. As we will see below, the set of controlling subgroups of I forms an upwards cone. Note that, since π_H is a $K[H]$-bimodule map, it follows that $\pi_H(I) \triangleleft K[H]$.

Lemma 1.1. [P9] *Let I be a two-sided ideal of $K[G]$.*

i. *$H \triangleleft G$ controls I if and only if $\pi_H(I) \subseteq I$. Indeed, when this occurs, then $I \cap K[H] = \pi_H(I)$ so $I = \pi_H(I)\cdot K[G]$.*
ii. *There exists a unique normal subgroup $\mathcal{C}(I)$, the controller of I, such that $H \triangleleft G$ controls I if and only if $H \supseteq \mathcal{C}(I)$.*

Proof. (i) If T is a right transversal for H in G, then it is easy to see that $\alpha \in K[G]$ can be written as the finite sum $\alpha = \sum_{t\in T} \pi_H(\alpha t^{-1})t$. Consequently,

$$(I \cap K[H])\cdot K[G] \subseteq I \subseteq \pi_H(I)\cdot K[G].$$

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In particular, if $\pi_H(I) \subseteq I$, then it is clear that $\pi_H(I) = I \cap K[H]$, so equality occurs above and H controls I. On the other hand, if H controls I, then

$$I = (I \cap K[H]) \cdot K[G] = \oplus \sum_{t \in T} (I \cap K[H])t,$$

and it follows that $\pi_H(I) = I \cap K[H] \subseteq I$.

(ii) Now let $\mathcal{H}$ denote the set of all normal subgroups H of G which control I, and define $C = \bigcap_{H \in \mathcal{H}} H$. We show by induction on $|\operatorname{supp} \alpha|$ that if $\alpha \in I$, then $\pi_C(\alpha) \in I$. To this end, let $\alpha \in I$ and assume that the result holds for all elements of I of smaller support size. If there exists $H \in \mathcal{H}$ with $|\operatorname{supp} \pi_H(\alpha)| < |\operatorname{supp} \alpha|$, then $\pi_H(\alpha) \in I$ by (i) above, so induction implies that $\pi_C(\alpha) = \pi_C(\pi_H(\alpha)) \in I$ since $H \supseteq C$. On the other hand, if $|\operatorname{supp} \pi_H(\alpha)| = |\operatorname{supp} \alpha|$ for all such H, then $\operatorname{supp} \alpha \subseteq \bigcap_{H \in \mathcal{H}} H = C$ and hence $\pi_C(\alpha) = \alpha \in I$. We conclude therefore that $\pi_C(I) \subseteq I$, so C controls I by (i) again, and consequently C is the unique minimal member of $\mathcal{H}$. $\square$

Thus, at the very least, our goal should be to understand the controller subgroup C of the Jacobson radical $\mathcal{J}K[G]$. Furthermore, we hope that this characteristic subgroup C of G is sufficiently small so that we can precisely describe the intersection ideal $\mathcal{J}K[G] \cap K[C] \triangleleft K[C]$.

As an example, define the *nilpotent radical* $\mathcal{N}R$ of any ring R to be the join of all its nilpotent ideals. Obviously, $\mathcal{N}R$ is a characteristic nil ideal of R and hence it is contained in $\mathcal{J}R$. As will be apparent, the structure of $\mathcal{N}K[G]$ is reasonably well understood. Indeed, if $\operatorname{char} K = 0$, then $\mathcal{N}K[G] = 0$, so it suffices to assume that $\operatorname{char} K = p > 0$. Note that if H is any group, then we use H^p to denote the subgroup of H generated by all its p-elements. In particular, if H is finite, then $H^p = \mathbb{O}^{p'}(H)$. The following result is proved using a powerful coset counting argument known as the Δ-*method*.

Theorem 1.2. [P1], [P2] *Let $\mathcal{D}^p(G)$ denote the set of all finite normal subgroups of G generated by their p-elements, and let $\Delta^p = \Delta^p(G) = \langle D \mid D \in \mathcal{D}^p(G) \rangle$. If $\operatorname{char} K = p > 0$, then*

i. *$\Delta^p = \Delta^p(G)$ is the controller of $\mathcal{N}K[G]$, and $\mathcal{N}K[G] = \mathcal{J}K[\Delta^p] \cdot K[G]$.*
ii. *$\mathcal{J}K[\Delta^p] = \bigcup_{D \in \mathcal{D}^p(G)} \mathcal{J}K[D]$.*
iii. *$\mathcal{N}K[G] \neq 0$ if and only if there exists $1 \neq D \in \mathcal{D}^p(G)$ and hence if and only if $1 \neq \Delta^p(G)$.*
iv. *$\mathcal{N}K[G]$ is nilpotent if and only if $\Delta(G)^p$ is finite.*

Note that (i) asserts that $\mathcal{J}K[\Delta^p]$ is contained in $\mathcal{N}K[G]$ and that it generates $\mathcal{N}K[G]$ as a right ideal. Furthermore, (iii) is an immediate consequence of parts (i) and (ii), along with the converse of Maschke's theorem. Obviously, the goal of the semiprimitivity problem should be to obtain a similar result for $\mathcal{J}K[G]$.

§2. Locally Finite Groups

Since it appears (see [A]) that all group algebras are semiprimitive in characteristic 0, we will assume throughout that $\operatorname{char} K = p > 0$. In some sense, paper [P6] split the semiprimitivity problem into two parts, namely the locally finite and the finitely generated cases (see Theorem 5.3). We begin with the first situation, and assume until further notice that G is a locally finite group. Observe that if $\alpha \in K[G]$, then the inclusion $\alpha \in \mathcal{J}K[G]$ is "local" in the following sense.

Lemma 2.1. *Suppose G is a locally finite group.*

i. *Let $\alpha \in K[G]$ and let $A = \langle \operatorname{supp} \alpha \rangle$ be its supporting subgroup. Then $\alpha \in \mathcal{J}K[G]$ if and only if $\alpha \in \mathcal{J}K[B]$ for all finite subgroups B with $A \subseteq B \subseteq G$.*
ii. *If $N \lhd\lhd G$, then $\mathcal{J}K[N] \subseteq \mathcal{J}K[G]$.*

In view of these two facts, we are naturally led to define the concept of a *locally subnormal subgroup*. Specifically, if A is a finite subgroup of G, then we write A lsn G provided $A \lhd\lhd B$ for all finite subgroups B of G with $A \subseteq B \subseteq G$. The previous lemma now clearly yields

Lemma 2.2. *If A lsn G, then $\mathcal{J}K[A] \subseteq \mathcal{J}K[G]$. In particular, if $1 \neq A = A^p$, then $\mathcal{J}K[G] \neq 0$.*

Thus, we obtain a fairly simple criterion which guarantees that $\mathcal{J}K[G] \neq 0$, and as it turns out, the converse is also true. Indeed, we have

Theorem 2.3. [P14], [P15] *Let $K[G]$ be the group algebra of the locally finite group G over a field K of characteristic $p > 0$. Then $\mathcal{J}K[G] \neq 0$ if and only if G has a nonidentity locally subnormal subgroup A generated by p-elements.*

The proof of this result is decidedly nontrivial. To start with, it depends upon a good deal of preliminary material and ultimately upon the Classification of the Finite Simple Groups as described in [G]. The basic ingredients include:

(1) Δ methods and the structure of $\mathcal{N}K[G]$.
(2) The semiprimitivity results in [P10] on locally finite, locally solvable groups.
(3) An argument of Formanek [F] on the finitary symmetric group FSym_∞.
(4) Kegel's lemma [Ke] which describes any locally finite simple group as a *limit of an approximating sequence of finite simple groups*.
(5) Work with Zalesskiĭ [PZ] on the semiprimitivity problem for locally finite, nonlinear, infinite simple groups.
(6) The characterization in [Be], [Bo], [HS] and [T] of locally finite, infinite simple, linear groups as groups of Lie type over locally finite fields.
(7) The work in [P12] on the semiprimitivity problem for locally finite, infinite simple, linear groups.
(8) Wielandt's result in [W1] and [W2] that the only primitive, infinite, finitary permutation groups are FSym_∞ and FAlt_∞ in their natural action.
(9) J. Hall's major results in [H1], [H2] and [H3] which show that the only nonlinear, locally finite, simple finitary linear groups are the infinite-dimensional classical groups.
(10) The work in [P11] and [P13] on the semiprimitivity of $K[G]$ for certain p'-f.c. covers of infinite simple, locally finite groups.
(11) An ultraproduct argument which requires the precise affirmative solution of the Schreier conjecture on the outer automorphism groups of the finite simple groups.

Note that an *f.c. group* is a group all of whose conjugacy classes are finite. In the context of locally finite groups, this is the same as being *locally normal*, that is, generated by finite normal subgroups. Now, when proving results about finite groups, one can frequently reduce the problem to a study of the composition factors and hence to the cases of solvable groups and nonabelian simple groups. On the other hand, this phenomenon rarely occurs when the groups being considered are infinite. Thus, it is quite surprising here that the proof of Theorem 2.3 eventually reduces to the locally solvable and the infinite simple cases.

§3. The Jacobson Radical

We continue with the assumption that G is locally finite. As is well known, every subgroup of a finite nilpotent group is subnormal. Hence if P is a locally finite p-group, then all its finite subgroups are locally subnormal. Furthermore, by the transitivity of subnormality, it follows that every finite subgroup of $\mathbb{O}_p(G)$ is locally subnormal in G. When $\mathbb{O}_p(G)=1$, the main result on $\mathcal{J}K[G]$ is

Theorem 3.1. [P16] *Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p>0$. Let $\mathcal{S}^p(G)$ denote the set of all locally subnormal subgroups A of G with $A=A^p$ and define $\mathbb{S}^p(G)=\langle A \mid A\in\mathcal{S}^p(G)\rangle$. If $\mathbb{O}_p(G)=1$, then we have*

- i. $S^p=\mathbb{S}^p(G)$ *is the controller of* $\mathcal{J}K[G]$ *and* $\mathcal{J}K[G]=\mathcal{J}K[S^p]\cdot K[G]$.
- ii. $\mathcal{J}K[S^p]=\bigcup_{A\in\mathcal{S}^p(G)}\mathcal{J}K[A]$.
- iii. $\mathcal{J}K[G]\neq 0$ *if and only if there exists* $1\neq A\in\mathcal{S}^p(G)$ *and hence if and only if* $\mathbb{S}^p(G)\neq 1$.

Obviously, this is the analog of Theorem 1.2 we had hoped for. The proof here is again quite complicated. For example, it uses Theorem 2.3 and many of the preliminary ingredients in the proof of that result. In addition, it requires:

1. (1) An argument of Dyment and Zalesskiĭ in [DZ].
2. (2) The main result of [P8] which asserts that $\mathcal{J}K[\mathbb{S}^p(G)]\cdot K[G]$ is a semiprime ideal of $K[G]$ when $\mathbb{O}_p(G)=1$.
3. (3) Work in [P7] on the structure of the characteristic subgroup $\mathbb{S}^p(G)$ (see Theorem 4.1).
4. (4) The result of Phillips in [Ph1] and [Ph2] which characterizes locally finite, primitive, finitary linear groups as solvable extensions of simple groups.
5. (5) Twisted group ring and crossed product techniques.

In other words, the proof of Theorem 3.1 essentially uses every result ever obtained on the semiprimitivity problem for group rings of locally finite groups.

The last part of the proof is rather technical, and is concerned with the following lifting problem. Suppose that $\mathbb{O}_p(G)=1$, $C\triangleleft G$, G/C is infinite simple and $\mathcal{J}K[C]=\mathcal{J}K[\mathbb{S}^p(C)]\cdot K[C]$. Then the goal is to show that $\mathcal{J}K[G]$ also satisfies the conclusion of the theorem. Now it turns out that if $G/C\not\cong \mathrm{FAlt}_\infty$, then G/C cannot act in an imprimitive fashion as automorphisms on a locally normal group. With this, and a certain amount of group theory, the problem reduces to the earlier results, contained in [P13], on p'-f.c. covers of simple groups. On the other hand, the finitary alternating group case requires much more serious considerations and is a rather painful task. Indeed, after a good deal of group theoretic work, combining all the imprimitive representations of G/C on the locally normal composition factors of G, we obtain the following *subcritical structure*.

- i. G has normal subgroups $D\subseteq X\subseteq L\subseteq C$ with $G/C=\mathrm{FAlt}_{\mathcal{I}}$ for some infinite set $\mathcal{I}$.
- ii. $L=C^p$, so that C/L is a p'-group.
- iii. D is a finite abelian p'-group which is central in G^p.
- iv. L is an f.c. group, and L/X is an abelian p-group.
- v. There exist finite normal subgroups X_i of C, for all $i\in\mathcal{I}$, with $(X_i)^g=X_{ig}$ where ig is the image of $i\in\mathcal{I}$ under the permutation $Cg\in\mathrm{FAlt}_{\mathcal{I}}$.
- vi. $D\subseteq X_i\subseteq X$ and X/D is the (weak) direct product $\prod_{i\in\mathcal{I}}(X_i/D)$.
- vii. $X_j/D\subseteq(\mathbb{C}_{L/D}(X_i/D))^p$ for all distinct $i,j\in\mathcal{I}$.

Unfortunately, this is as far as the group theory can go on its own. Since $\mathbb{S}^p(G) = L$ here, we must prove that $\mathcal{J}K[G] = \mathcal{J}K[L]\cdot K[G]$, and this is done by brute force using crossed product techniques.

Finally, very little additional work is needed to handle groups with $\mathbb{O}_p(G) \neq 1$. Indeed, the following is a fairly obvious generalization of the main result.

Corollary 3.2. [P16] *Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p > 0$. Let $P = \mathbb{O}_p(G)$ and let $T^p = \mathbb{T}^p(G) \supseteq P$ be defined by $T^p/P = \mathbb{S}^p(G/P)$. If I is the kernel of the natural epimorphism $K[T^p] \to K[T^p/P] = K[\mathbb{S}^p(G/P)]$, then*

i. *$T^p = \mathbb{T}^p(G)$ is the controller of $\mathcal{J}K[G]$, and $\mathcal{J}K[G] = \mathcal{J}K[T^p]\cdot K[G]$.*
ii. *$\mathcal{J}K[T^p] \supseteq I$ and $\mathcal{J}K[T^p]/I = \mathcal{J}K[\mathbb{S}^p(G/P)]$.*
iii. *$\mathcal{J}K[G] \neq 0$ if and only if there exists a nonidentity locally subnormal subgroup A of G with $A = A^p$.*

Of course, since $\mathbb{O}_p(G/P) = 1$ in the above, $\mathcal{J}K[\mathbb{S}^p(G/P)]$ can be satisfactorily described using Theorem 3.1(ii).

§4. The Structure of $\mathbb{S}^p(G)$

We now take a closer look at $\mathbb{S}^p(G)$, again assuming that G is locally finite. Recall that $\mathcal{S}^p(G)$ denotes the set of all locally subnormal subgroups A of G with $A = A^p$, and that $\mathbb{S}^p(G) = \langle A \mid A \in \mathcal{S}^p(G)\rangle$. Furthermore, for any such A as above, let $\operatorname{len} A$ denote the composition length of A, namely the common length of all composition series for the group. Since A is finite, $\operatorname{len} A$ is certainly finite.

Theorem 4.1. [P7] *Let G be a locally finite group with $\mathbb{O}_p(G) = 1$ and, for any integer $n \geq 1$, let*

$$\mathbb{S}^p_n(G) = \langle A \mid A \in \mathcal{S}^p(G) \text{ and } \operatorname{len} A \leq n\rangle.$$

Then $\mathbb{S}^p(G)$ is the ascending union of its characteristic f.c. subgroups $\mathbb{S}^p_n(G)$.

Now suppose, in the above situation, that A lsn G, $A = A^p$, and say $\operatorname{len} A = n$. Then $A \subseteq \mathbb{S}^p_n(G)$, and the latter is a normal f.c. subgroup of G. In particular, since $\mathbb{S}^p_n(G)$ is generated by its finite normal subgroups, there exists such a subgroup B with $A \subseteq B \triangleleft \mathbb{S}^p_n(G)$. But $|B| < \infty$, so $A \triangleleft\triangleleft B$ and therefore $A \triangleleft\triangleleft G$. Furthermore, if we take B to be the normal closure of A in $\mathbb{S}^p_n(G)$, then $B = B^p$ and $B \triangleleft\triangleleft G$ with subnormal depth at most 2. As a consequence, these several concepts all merge into one. In particular, it follows from Corollary 3.2 that $\mathcal{J}K[G] \neq 0$ if and only if either $\mathbb{O}_p(G) \neq 1$ or G has a finite nonidentity subgroup $A = A^p$ of subnormal depth ≤ 2. Thus we obtain a somewhat simpler criterion for the semiprimitivity of $K[G]$, and clearly

$$\mathcal{J}K[\mathbb{S}^p(G)] = \bigcup_{n=1}^{\infty} \mathcal{J}K[\mathbb{S}^p_n(G)].$$

At this point, it is appropriate to consider some examples associated with the preceding results. For instance, we might ask whether Theorem 4.1 is best possible or whether $\mathbb{S}^p(G)$ is necessarily always an f.c. group. In addition, it is clear that $\mathbb{T}^p(G) \supseteq \mathbb{S}^p(G)$, and it is natural to ask whether these subgroups can be different. The answers here are all contained in

Lemma 4.2. *Let p be a fixed prime.*

i. *There exists a p-group P and an element $x \in P$ such that the normal closure $\langle x\rangle^P$ is not an f.c. group.*
ii. *There is a group G with $\mathbb{O}_p(G) = 1$ and with $\mathbb{S}^p(G)$ not an f.c. group.*
iii. *There exists a group G with $\mathbb{T}^p(G)$ strictly larger than $\mathbb{S}^p(G)$.*

Proof. (i) Set $P = A \rtimes \langle x\rangle$, where A is an abelian p-group and x is an element of order p^2 which acts on A with $[A, x, x]$ infinite. For example, we could take P to be the wreath product $P = B \wr \langle x\rangle$ with B an infinite abelian p-group. Finally observe that $Q = \langle x\rangle^P = \langle x, [A, x]\rangle$ and note that Q is not an f.c. group since $[Q, x] \supseteq [A, x, x]$ is infinite.

(ii) Now let P be a transitive p-subgroup of FSym_∞. For example, we could take P to be the infinite wreath product $P = \dots(((Z_p \wr Z_p) \wr Z_p) \wr Z_p)\dots$ of cyclic groups Z_p of order p. Furthermore, let X be a finite nonabelian simple group of order divisible by p and define G to be the permutation wreath product $G = X \wr P$. We claim that G is a locally finite group with $\mathbb{O}_p(G) = 1$, $G = \mathbb{S}^p(G)$ and with G not an f.c. group.

To start with, G is the semidirect product $G = Y \rtimes P$ where Y is the (weak) direct product $Y = \prod_{i=1}^{\infty} X_i$ of countably many copies X_i of X. Thus Y is a semisimple normal subgroup of G which is the kernel of the conjugation permutation action of G on the set $\mathcal{X} = \{ X_i \mid i = 1, 2, \dots\}$. Since $\mathbb{O}_p(G) \cap Y = 1$, it follows that $\mathbb{O}_p(G)$ acts trivially on $\mathcal{X}$, so $\mathbb{O}_p(G) \subseteq Y$ and hence $\mathbb{O}_p(G) = 1$.

Next, it is clear that $\mathbb{S}^p(G) \supseteq Y$. Furthermore, let Q be any finite subgroup of P and let Z be the direct product of the finitely many X_is which are moved by Q. We claim that $A = ZQ$ is a locally subnormal subgroup of G. To this end, let B be any finite subgroup of G which contains A. Since Q centralizes the X_i factors of Y not in Z, it is clear that $A \triangleleft YQ$. Furthermore, since $G/Y \cong P$ is a locally finite p-group, it follows that $YQ/Y \triangleleft\triangleleft YB/Y$ and hence $YQ \triangleleft\triangleleft YB$. In other words, we have shown that $A \triangleleft\triangleleft YB$ and hence $A \triangleleft\triangleleft B$, as required. Thus $Q \subseteq \mathbb{S}^p(G)$ and, since $Q \subseteq P$ is arbitrary, we have $\mathbb{S}^p(G) = G$.

Finally, note that P has no nontrivial finite normal subgroups. Indeed, if $N \neq 1$ were such a subgroup, then N moves only finitely many points $\Omega \neq \emptyset$ under the permutation action of $P \subseteq \mathrm{FSym}_\infty$. Furthermore, since $N \triangleleft P$, P must act on Ω, and this contradicts the fact that P is transitive on the infinite set $\{1, 2, 3, \dots\}$. Thus, no such N exists and, in particular, P cannot be an f.c. group. Hence, since $G \supseteq P$, we conclude that G is also not an f.c. group.

(iii) We continue with the notation of part (ii). In addition, we let D be an infinite abelian p-group, and we set $H = D \wr G = E \rtimes G$ where E is the (weak) direct product $E = \prod_{g\in G} D_g$ of copies D_g of D indexed by the elements of G. Since $\mathbb{O}_p(G) = 1$, it is clear that $E = \mathbb{O}_p(H)$. Furthermore, since $H/\mathbb{O}_p(H) \cong G$ and $G = \mathbb{S}^p(G)$, it follows that $H = \mathbb{T}^p(H)$.

It remains to show that $H \neq \mathbb{S}^p(H)$ and indeed we show that $\mathbb{S}^p(H) \subseteq \mathbb{O}_p(H) = E$. Suppose, by way of contradiction, that this is not the case. Then there must exist a locally subnormal subgroup A of H which is not a p-group. Since $H/EY \cong P$ is a p-group and A is not a p-group, it follows that $A \cap EY$ is not a p-group. Thus, without loss of generality, we may assume that $A \subseteq EY$ and $A \not\subseteq E$. Note that A lsn EY and hence EA/E is a locally subnormal subgroup of the semisimple group $EY/E \cong Y$. Consequently, $EA/E \cong A/(E \cap A)$ is a finite semisimple group.

Set $B = E \cap A$ and note that $B \triangleleft EA$ since E is abelian. Thus, since A lsn EA, it follows that (A/B) lsn (EA/B). But $A/B = A/(E \cap A)$ is semisimple, so its normal closure in EA/B is also semisimple and hence disjoint from the normal p-subgroup E/B. It follows that A/B centralizes E/B and hence that $[E, A] = [E, A/B] \subseteq B$ is finite. But $E = \prod_{g \in G} D_g$ and $A/B \neq 1$ permutes the factors D_g in a nontrivial manner. Thus, $[E, A/B]$ must project onto one of the D_gs, and consequently, this commutator group is infinite, a contradiction. We conclude therefore that all locally subnormal subgroups of H are p-groups, so $\mathbb{S}^p(H) \subseteq \mathbb{O}_p(H) \neq H = \mathbb{T}^p(H)$. $\square$

In particular, the assumption that $\mathbb{O}_p(G) = 1$ is required in Theorem 4.1 and even with this, $\mathbb{S}^p(G)$ need not be an f.c. group. Furthermore, the group $\mathbb{T}^p(G)$ cannot be replaced by $\mathbb{S}^p(G)$ in Corollary 3.2.

§5. Finitely Generated Groups

As usual, we suppose that K is a field of characteristic $p > 0$, but now we consider the semiprimitivity problem for finitely generated groups. The guess here is

Conjecture 5.1. *If G is a finitely generated group, then $\mathcal{J}K[G] = \mathcal{N}K[G]$.*

This would be a wonderful result for two reasons. First, in view of Theorem 1.2, we would have an adequate description of $\mathcal{J}K[G]$ for such groups. But more importantly, since an arbitrary group has its finitely generated subgroups as a local system, we could use this result to describe $\mathcal{J}K[G]$ in general. Specifically, let G be an arbitrary group and let $\Lambda^+(G)$ be generated by all those finite subgroups A of G whose normal closure A^H is finite for every finitely generated subgroup H of G. Then, we have

Lemma 5.2. [P6] *Let G be an arbitrary group.*

i. *$\Lambda^+(G)$ is a locally finite characteristic subgroup of G.*
ii. *If $N \triangleleft G$ with $N \subseteq \Lambda^+(G)$, then $\Lambda^+(G/N) = \Lambda^+(G)/N$.*

Part (ii) above indicates that Λ^+ is a "radical" for groups, somewhat of a surprise. Note that, if G is locally finite, then $\Lambda^+(G) = G$. Thus part (i) above is best possible in that $\Lambda^+(G)$ can be any locally finite group. In any case, because of Corollary 3.2, we know the precise structure of $\mathcal{J}K[\Lambda^+(G)]$ and, by using that result and Theorem 1.2, we can now prove

Theorem 5.3. [P6], [P16] *Let $K[G]$ be the group algebra of an arbitrary group G over a field K of characteristic $p > 0$. Assume that, for each finitely generated subgroup H of G, we have $\mathcal{J}K[H] = \mathcal{N}K[H]$. Then*

i. *$\mathcal{J}K[G] = \mathcal{J}K[L^+]\cdot K[G]$ where $L^+ = \Lambda^+(G)$.*
ii. *The controller of $\mathcal{J}K[G]$ is $T^p = \mathbb{T}^p(L^+)$ and $\mathcal{J}K[G] = \mathcal{J}K[T^p]\cdot K[G]$.*
iii. *$\mathcal{J}K[G] \neq 0$ if and only if $T^p \neq 1$.*

Thus it remains to prove Conjecture 5.1, but unfortunately this seems to be a rather hopeless task at the moment. Nevertheless, we do know that the conjecture holds when G is a solvable group (see [HP], [P3], [Z1] and [Z2]), and also when G is a linear group (see [P4], [P5] and [P6]). Furthermore, we have

Theorem 5.4. [P6] *If G is a finitely generated group, then $\mathcal{N}K[G]$ is a semiprime ideal of $K[G]$. In other words, $\mathcal{N}(K[G]/\mathcal{N}K[G]) = 0$.*

In particular, $\mathcal{N}$ behaves like a "radical" when applied to group algebras of finitely generated groups. On the other hand, the following example of Bergman shows that his behavior is really atypical.

Lemma 5.5. *There exists a finitely generated K-algebra R with $\mathcal{N}(R/\mathcal{N}R) \neq 0$.*

Proof. Let $F = K\langle x, y\rangle$ be the free K-algebra in the variables x and y, and let $I \subseteq F$ be the K-linear span of all monomials of the form $\ldots xy^i x \ldots xy^i x \ldots$ for any integer $i = 0, 1, 2, \ldots$. Then I is clearly an ideal of F and we set $R = F/I$. Note that if $\bar{x}$ and $\bar{y}$ denote the images of x and y in R, then R has as a K-basis all monomials in $\bar{x}$ and $\bar{y}$ not of the above form.

Now, for any integer $i \geq 0$, we have $R\bar{x}\bar{y}^i\bar{x}R\bar{x}\bar{y}^i\bar{x}R = 0$, so $R\bar{x}\bar{y}^i\bar{x}R$ is an ideal of R of square 0 and hence $R\bar{x}\bar{y}^i\bar{x}R \subseteq \mathcal{N}R$. Consequently, we have

$$R\bar{x}R\bar{x}R = \sum_{i=0}^{\infty} R\bar{x}\bar{y}^i\bar{x}R \subseteq \mathcal{N}R,$$

so $(R\bar{x}R)^2 \subseteq \mathcal{N}R$ and $\tilde{x} = \bar{x} + \mathcal{N}R \in \mathcal{N}(R/\mathcal{N}R)$. Finally, note that $\bar{x} \notin \mathcal{N}R$ since $(\bar{x}\bar{y})(\bar{x}\bar{y}^2)(\bar{x}\bar{y}^3)\cdots(\bar{x}\bar{y}^n) \neq 0$ for any integer $n \geq 1$, and hence $\tilde{x}$ is a nonzero element of $R/\mathcal{N}R$, as required. □

§6. Burnside Groups

Again, let K be a field of characteristic $p > 0$ and let G be a finitely generated group. If $\mathcal{A}K[G]$ denotes the *augmentation ideal* of $K[G]$, namely the kernel of the natural homomorphism $K[G] \to K[G/G] \cong K$, then $\mathcal{A}K[G]$ is a maximal right ideal of $K[G]$ and hence it contains the Jacobson radical $\mathcal{J}K[G]$. In [K1] and [K2], Kaplansky posed a number of interesting ring theoretic problems, including some on group algebras. One of the latter concerned the general semiprimitivity problem and one concerned the following special case.

Conjecture 6.1. *If $\operatorname{char} K = p > 0$ and G is a finitely generated group, then $\mathcal{J}K[G] = \mathcal{A}K[G]$ if and only if G is a finite p-group.*

We note that the equality $\mathcal{J}K[G] = \mathcal{A}K[G]$ really means that $K[G]$ has precisely one irreducible representation, namely the *principal representation* $\rho : K[G] \to K$ with $\rho(G) = 1$. Of course, if G is a finite p-group, then it is easy to see that $\mathcal{J}K[G] = \mathcal{A}K[G]$. Thus the real concern here is with the converse direction, and this is actually a consequence of Conjecture 5.1. To start with, we list a number of elementary properties associated with the augmentation ideal. For convenience, if I is any right ideal of $K[G]$, we write $\mathcal{G}(I) = \{\, g \in G \mid g - 1 \in I \,\}$. Clearly, $\mathcal{G}(I) = G$ if and only if $I \supseteq \mathcal{A}K[G]$.

Lemma 6.2. *Let $K[G]$ be an arbitrary group algebra and write $A = \mathcal{A}K[G]$.*

i. *If I is a right ideal of $K[G]$, then $\mathcal{G}(I)$ is a subgroup of G.*
ii. *If $G = \langle x_1, x_2, \ldots, x_n\rangle$ is finitely generated, then $A = \sum_{i=1}^n (x_i - 1)K[G]$ is a finitely generated right ideal.*
iii. *A has a nonzero right or left annihilator if and only if G is finite.*
iv. *$\mathcal{G}(A^2) \supseteq G' = [G, G]$, the commutator subgroup of G.*

Proof. (i) If $x, y \in \mathcal{G}(I)$, then $yx^{-1} - 1 = [(y-1) - (x-1)]x^{-1} \in I$, so $yx^{-1} \in \mathcal{G}(I)$.

(ii) Note that $I = \sum_{i=1}^n (x_i - 1)K[G]$ is a right ideal of $K[G]$ contained in A. On the other hand, since $\mathcal{G}(I)$ is a group, it follows that $\mathcal{G}(I) \supseteq \langle x_1, x_2, \ldots, x_n\rangle = G$, and consequently $I \supseteq A$.

(iii) Say $0 \neq \alpha \in K[G]$ with $\alpha A = 0$. Then $\alpha(g - 1) = 0$ for all $g \in G$, so $\alpha = \alpha g$ and $\operatorname{supp}\alpha = (\operatorname{supp}\alpha)g$. Since $\operatorname{supp}\alpha$ is closed under right multiplication by G, it follows that $G = \operatorname{supp}\alpha$ is finite. On the other hand, if G is finite, then $\beta = \sum_{g\in G} g$ annihilates A on both sides.

(iv) If $x, y \in G$, then

$$\begin{aligned} xyx^{-1}y^{-1} - 1 &= (xy - yx)x^{-1}y^{-1} \\ &= [(x-1)(y-1) - (y-1)(x-1)]x^{-1}y^{-1} \end{aligned}$$

is contained in A^2, and consequently $\mathcal{G}(A^2)$ contains all commutators. □

Now suppose that G is a finitely generated group with $A = \mathcal{A}K[G] = \mathcal{J}K[G]$. If we assume the validity of Conjecture 5.1, then $A = \mathcal{N}K[G]$ and, since A is a finitely generated ideal by Lemma 6.2(ii), this implies that A is nilpotent. Thus A has a nonzero right and left annihilator, so G is finite by the previous lemma, and it follows easily (see Theorem 6.3(i)) that G is a p-group. In other words, the solution to Kaplansky's problem is an immediate consequence of Conjecture 5.1.

The following result lists essentially all we know of a general nature about Conjecture 6.1. For the most part, it is based on the work of Lichtman in [L].

Theorem 6.3. *Let G be a finitely generated group and let $\operatorname{char} K = p > 0$. If $\mathcal{J}K[G] = \mathcal{A}K[G]$, then*

i. *G is a p-group.*
ii. [L] *If $G \neq 1$, then $G \neq G'$.*
iii. [L] *If G is an infinite group, then G has an infinite, residually finite homomorphic image.*
iv. *Any maximal subgroup of G is normal of index p.*
v. *If H is a subgroup of G of infinite index, then there exists a chain of subgroups $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq H$ with $G_{i+1} \triangleleft G_i$ and $|G_i/G_{i+1}| = p$.*

Proof. If L is a subgroup of G, then $K[G] = K[L] \oplus U$, where U is the kernel of the projection map π_L. Thus, we have a $K[L]$-bimodule decomposition of $K[G]$, and it follows easily that $\mathcal{J}K[G] \cap K[L] \subseteq \mathcal{J}K[L]$. In particular, since $\mathcal{J}K[G] = \mathcal{A}K[G]$, we conclude that $\mathcal{J}K[L] = \mathcal{A}K[L]$.

(i) Let $x \in G$ and set $X = \langle x\rangle$. Then, by the above, $\mathcal{J}K[X] = \mathcal{A}K[X]$, and it follows that x must have finite order. But then $\mathcal{J}K[X] = \mathcal{A}K[X]$ is nilpotent so, for some integer $n \geq 1$, we have $0 = (x-1)^{p^n} = x^{p^n} - 1$, and G is a p-group.

(ii), (iv) Let H be a proper subgroup of G, let $A = \mathcal{A}K[G] = \mathcal{J}K[G]$ and set $B = \mathcal{A}K[H]\cdot K[G]$. Then $V = A/B$ is a nonzero finitely generated right $K[G]$-module, by Lemma 6.2(ii), and hence Nakayama's lemma implies that $V \supsetneq V\cdot\mathcal{J}K[G] = V\cdot A$. In other words, $A \supsetneq B + A^2$. Now $\mathcal{G}(B) \supseteq H$ and $\mathcal{G}(A^2) \supseteq G'$, by Lemma 6.2(iv). Thus $\mathcal{G}(B + A^2) \supseteq HG'$, by Lemma 6.2(i), and since this right ideal is properly smaller than $A = \mathcal{A}K[G]$, it follows that $HG' \subseteq \mathcal{G}(B + A^2) \subsetneq G$.

If $G \neq 1$, we can take $H = 1$ and deduce that $G' \subsetneq G$. This yields (ii). On the other hand, if H is a maximal subgroup of G, then since $H \subseteq HG' \subsetneq G$, we conclude that $H \supseteq G'$. Thus $H \triangleleft G$ and then clearly, $|G/H| = p$.

(iii) Suppose now that G is infinite and let L denote the intersection of all normal subgroups of G of finite index. Then G/L is a residually finite homomorphic image of G and it remains to show that this factor group is infinite. To this end, suppose that $|G : L| < \infty$. Then L is also finitely generated and $L \neq 1$. Part (ii) now implies that $L \supsetneq L'$, and thus L' is a normal subgroup of finite index in G properly smaller than L. Since this contradicts the definition of L, we conclude that $|G/L| = \infty$, as required.

(v) Finally, let $|G : H| = \infty$. We prove by induction on n that there exists a suitable chain of subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq H$ with $|G_i/G_{i+1}| = p$. To this end, assume that we have found the group G_n. Then $|G : G_n| = p^n < \infty$, so G_n is finitely generated and $G_n \supsetneq H$. Therefore, we can let G_{n+1} be a maximal subgroup of G_n which contains H. By (iii) above, we know that $G_{n+1} \triangleleft G_n$ with factor group cyclic of order p. Thus G_{n+1} exists, and the result follows. □

It is not surprising that maximal subgroups come into play here. Indeed, if $K = \mathrm{GF}(p)$ and if every maximal subgroup of the wreath product $Z_p \wr G$ is normal, then $\mathcal{J}K[G] = \mathcal{A}K[G]$. To see this, note that $Z_p \wr G = V \rtimes G$ where V, written additively, is isomorphic to the right regular $K[G]$-module. In particular, if W is any irreducible $K[G]$-module, then $W \cong V/M$ for some maximal submodule M, and then it is clear that MG is a maximal subgroup of VG. But if all maximal subgroups are assumed to be normal, then $[V,G] \subseteq V \cap MG = M$ and hence G acts trivially on $V/M \cong W$. In other words, W can only be the principal module.

Obviously, finitely generated, infinite p-groups are related to the Burnside problem. If G is a Tarski monster, as constructed by Ol'shanskiĭ in [O], then G is simple, so (ii) implies that $\mathcal{J}K[G] \neq \mathcal{A}K[G]$. If G has bounded period, as first constructed by Novikov and Adjan in [NA1], [NA2] and [NA3], then the affirmative solution of the restricted Burnside problem by Zelmanov in [Ze1] and [Ze2] implies that G cannot have an infinite, residually finite, homomorphic image. Thus, (iii) implies that $\mathcal{J}K[G] \neq \mathcal{A}K[G]$. On the other hand, the groups constructed by Golod in [Go] and [GoS] and by Gupta and Sidki in [GuS] and [S1] are infinite, residually finite, p-groups, so parts (ii) and (iii) cannot help here. Furthermore, there are unfortunately no known results on the maximal subgroups of such groups, so we do not know whether parts (iv) or (v) will help either.

Now if G is a Golod group defined over the field K, then G is a subgroup of the unit group of a K-algebra R, and there is an epimorphism $\theta : K[G] \to R$ with $\theta(\mathcal{A}K[G])$ an infinite dimensional nil ideal. Fortunately, Siderov [Si] has shown that θ is not an isomorphism at least when the construction parameters satisfy certain natural conditions. Furthermore, Sidki [S2] introduced a ring theoretic variant of the recursive methods used to construct the Gupta-Sidki groups, and he showed that these groups have at least one nonprincipal irreducible respresentation in characteristic p. Indeed, starting with this result, it was shown in [PT] that if K is sufficiently large, then the Gupta-Sidki groups must have infinitely many nonisomorphic irreducible representations.

Certainly, an affirmative solution to Kaplansky's problem would be a major step in the general semiprimitivity problem. Obviously, it will prove to be a decidedly nontrivial task. Riley [R] has suggested that there might be an approach here via pro-p groups. But, even with this idea, there are still fundamental difficulties which must be overcome.

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