

THE SEMIPRIMITIVITY PROBLEM FOR TWISTED GROUP ALGEBRAS OF LOCALLY FINITE GROUPS

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Dedicated to the memory of Professor Brian Hartley

Abstract. Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p > 0$. In this paper, we show that $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p . Thus we settle the semiprimitivity problem for such group algebras by verifying a conjecture which dates back to the middle 1970's. Of course, if G has a locally subnormal subgroup of order divisible by p , then it is easy to see that the Jacobson radical $\mathcal{J}K[G]$ is not zero. Thus, the real content of this problem is the converse statement. Our approach here builds upon a recent paper where we came tantalizingly close to a complete solution by showing that if G has no nonidentity locally subnormal subgroup, then $K[G]$ is semiprimitive. In addition, we use a two step process, suggested by certain earlier work on semiprimitivity, to complete the proof. The first step is to assume that all locally subnormal subgroups are central. Since this is easily seen to reduce to a twisted group algebra problem, our goal for this part is to show that $K^t[G]$ is semiprimitive when G has no nontrivial locally subnormal subgroup. In other words, we duplicate the work of the previous paper, but in the context of twisted group algebras. As it turns out, almost all of the techniques of that paper carry over directly to this new situation. Indeed, there are only two serious technical problems to overcome. The second step in the process requires that we deal with certain extensions by finitary groups, and here we use recent results on primitive, finitary linear groups to show that the factor groups in question have

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well behaved subnormal series. With this, we can apply previous machinery to handle the extension problem and thereby complete the proof of the main theorem.

§1. INTRODUCTION

The semiprimitivity problem for group algebras $K[G]$ was intensively studied in the 1970's and again in the 1990's (see [17]). In [11, Section I], the problem was essentially split into two parts, namely the finitely generated case and the locally finite case. The first part still appears to be a rather hopeless task, but the second part was always more promising. In [13], the last of the 1970's work on semiprimitivity, the case of locally finite, locally p -solvable groups was settled. At that time, it appeared that the Classification of Finite Simple Groups (CFSG) would be required to deal with more general locally finite groups, and this has indeed proved to be the case. In the last few years, consequences of CFSG have been used in [14], [15], [16], and [19] to study the semiprimitivity of group algebras of locally finite groups under certain global assumptions. Furthermore, these results were then used in [18] to deal with the more important local assumptions.

For the remainder of this paper, G will always denote a locally finite group and K will be a field of characteristic $p > 0$. As usual, $K[G]$ is the *group algebra* of G over K , and this group algebra is *semiprimitive* when its Jacobson radical $\mathcal{JK}[G]$ is zero. Recall that a finite subgroup A of G is said to be *locally subnormal* in G , written $A \text{ lsn } G$, if A is subnormal in every finite subgroup B with $A \subseteq B \subseteq G$. For example, if G is locally nilpotent, then every finite subgroup of G is locally subnormal. Furthermore, every finite subnormal subgroup of G is certainly locally subnormal. The following is [18, Lemma 2.1].

Lemma 1.1. *If A is a subgroup of G which is either subnormal or locally subnormal, then $\mathcal{JK}[A] \subseteq \mathcal{JK}[G]$.*

In particular, if $A \text{ lsn } G$ with $|A|$ divisible by p , then $K[G]$ is not semiprimitive. Surprisingly, this simple observation seemed to characterize those group algebras with $\mathcal{JK}[G] \neq 0$ and, in the middle 1970's, we conjectured that $K[G]$ is semiprimitive if and only if no such A exists. In fact, this was verified for locally p -solvable groups in [13], and paper [18] came close to a complete solution by proving

Theorem 1.2. *If G has no nonidentity locally subnormal subgroup, then $K[G]$ is semiprimitive.*

Obviously this is very suggestive, but there is still more to be done to validate the conjecture. Based on the approach taken in [13], the next step in the process is to study groups in which all locally subnormal subgroups are central, and that is the goal of first part of this paper. Indeed, we prove

Theorem 1.3. *Let G be a locally finite group in which all locally subnormal subgroups are central, and let K be a field of characteristic $p > 0$. If $Z = \mathbb{Z}(G)$ and P*

is the p -primary part of the center Z , then

$$\mathcal{J}K[G] = \mathcal{J}K[Z] \cdot K[G] = \mathcal{J}K[P] \cdot K[G]$$

where $\mathcal{J}K[P]$ is the augmentation ideal of $K[P]$, namely the kernel of the augmentation map $K[P] \rightarrow K$. In particular, $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p .

As we will see, this is an immediate consequence of

Theorem 1.4. *Let G be a locally finite group with no nonidentity locally subnormal subgroup, and let K be a field of characteristic $p > 0$. Then any twisted group algebra $K^t[G]$ is semiprimitive.*

In other words, our task for this part is to reprove Theorem 1.2 in the context of twisted group algebras, and fortunately almost all of the techniques used in [18] carry over to this new situation. However, there are two places where considerable additional effort is required. First, paper [18] uses the earlier work of [13] which considered ordinary group algebras of locally p -solvable groups. Therefore, the latter results must be extended to the twisted case, and to do this, the Delta methods of [13] have to be reformulated with somewhat more care.

The second problem is more serious and more subtle; one might easily miss it on a first reading. In [18, Section 6], a certain group algebra $K[S]$ is mapped homomorphically onto $K[\bar{S}]$, where $\bar{S} = S/\text{sol } S$ is the top semisimple layer of the finite group S . Of course, such homomorphisms do not exist in general for twisted group algebras. One possible fix for this problem is to use the generalized Fitting subgroup \mathbb{F}^* (see [2, Definition 1.26]) throughout the entire proof of Theorem 1.4, because the semisimple layer of \mathbb{F}^* occurs as a subgroup. However it turns out that \mathbb{F}^* is just not large enough for our purposes. So we continue with our original approach, but now we borrow certain results from [13] to help control the action of p -elements on p -subgroups. Since this is obviously of no benefit when the simple factors are p' -groups, we are therefore forced to absorb these p' -groups into $\text{sol } S$. In other words, we replace the solvable radical of S by the an appropriate p -solvable radical. Hopefully, this will all become clear as we proceed.

The proofs of Theorems 1.3 and 1.4 start in the next section and occupy most of the remainder of this paper. Once they are completed, we study finitary linear groups which are the normal closure of a finite subgroup. Specifically, we show that such groups have a finite subnormal series with factors which are either infinite simple, locally solvable, or f.c. groups. This result, along with certain previously developed machinery, then allows us to handle the extension problem, the second step in the process suggested by [13]. Finally, in Section 9, we prove our main result, namely

Theorem 1.5. *Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p > 0$. Then $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p .*

Obviously, this theorem settles the semiprimitivity problem for group algebras of locally finite groups. However, it does not yield a description of the Jacobson radical $\mathcal{JK}[G]$ when G has locally subnormal subgroups of order divisible by p . This latter task will surely require some additional ideas, and therefore it is best left for a latter project. We close this section with a personal note.

Brian Hartley and I have been good friends since he visited me at the University of Wisconsin twenty years ago. Two years later, I visited him at the Mathematics Institute of Warwick University and it was there that [13] was written. As acknowledged in that paper, Brian's numerous helpful suggestions had an important impact on the work. Brian's contribution to this paper is of a different nature. He was one of the main organizers of the Conference on Finite and Locally Finite Groups which was held this past summer at the University of Bogazici in Istanbul, Turkey. Brian kindly invited me to attend and, because of that, I was able to learn about several techniques which are now used in this paper and in [18]. Indeed, a rough outline of [18] was actually written at that conference. Throughout his career, Brian Hartley was a stellar researcher, a generous colleague, and an influential leader in the study of infinite groups and group rings. His recent, untimely death at the age of 55 was a tragedy. This paper is dedicated to his memory.

§2. DELTA METHODS

Let G be an arbitrary multiplicative group and let K be a field. Then a *twisted group algebra* $K^t[G]$ is an associative K -algebra with K -basis \bar{G} , a copy of G , and with multiplication defined distributively using

$$\bar{x}\bar{y} = t(x, y)\bar{xy} \quad \text{for all } x, y \in G.$$

Here $t: G \times G \rightarrow K^\bullet$ is the *twisting function* and associativity is easily seen to be equivalent to

$$t(x, y)t(xy, z) = t(x, yz)t(y, z) \quad \text{for all } x, y, z \in G.$$

In other words, t must be a 2-cocycle. On the other hand, a simple (diagonal) change of basis, replacing each \bar{x} by $\tilde{x} = d(x)\bar{x}$ with $d: G \rightarrow K^\bullet$, obviously maintains the same structure but replaces the twisting t by

$$\tilde{t}(x, y) = d(x)d(y)d(xy)^{-1}t(x, y).$$

Thus t and \tilde{t} are equivalent modulo a 2-coboundary and therefore the various twisted group algebras $K^t[G]$ are in one-to-one correspondence with the elements of the cohomology group $H^2(G, K^\bullet)$. By way of a diagonal change of basis, we can assume without loss of generality that $\bar{1} = 1$ is the identity element of $K^t[G]$. Furthermore, $\mathcal{G} = \{k\bar{g} \mid k \in K^\bullet, g \in G\}$ is a group of units of $K^t[G]$, the so-called *group of trivial units*, and $\mathcal{G}/K^\bullet \cong G$.

If H is a subgroup of G , then $K^t[H]$, the K -linear span of \bar{H} in $K^t[G]$, is obviously a twisted group algebra of H . Furthermore, by using coset representatives for H in G , it is clear that $K^t[G]$ is a free right and left $K^t[H]$ -module. In particular, we have (see [10, Lemma 1.9])

Lemma 2.1. *If H is a subgroup of G , then $\mathcal{J}K^t[G] \cap K^t[H] \subseteq \mathcal{J}K^t[H]$. Thus $\mathcal{J}K^t[G]$ is a nil ideal, and $\mathcal{J}K^t[G] = 0$ when G is a p' -group.*

The following is a simple consequence of Nakayama's Lemma and of Maschke's Theorem (see [10, Propositions 1.3 and 1.5]).

Proposition 2.2. *Let H be a normal subgroup of G of finite index n . Then*

$$(\mathcal{J}K^t[G])^n \subseteq \mathcal{J}K^t[H] \cdot K^t[G] \subseteq \mathcal{J}K^t[G].$$

In particular, $\mathcal{J}K^t[G]$ is nilpotent if and only if $\mathcal{J}K^t[H]$ is nilpotent. Furthermore, if $1/n \in K$, then $\mathcal{J}K^t[G] = \mathcal{J}K^t[H] \cdot K^t[G]$.

With this, we can sketch a proof of the twisted analog of Lemma 1.1.

Lemma 2.3. *If A is a subgroup of G which is either subnormal or locally subnormal, then $\mathcal{J}K^t[A] \subseteq \mathcal{J}K^t[G]$.*

Proof. It suffices to assume that $A \triangleleft G$. Fix $\alpha \in \mathcal{J}K^t[A]$ and let $\beta \in K^t[G]$ be arbitrary. Then there exists a subgroup L of G with $L \supseteq A$, $|L/A| < \infty$ and $\beta \in K^t[L]$. By the previous proposition, $\alpha\beta \in \mathcal{J}K^t[L]$ and, in particular, $\alpha\beta$ is nilpotent. Thus $\alpha K^t[G]$ is a nil right ideal of $K^t[G]$ and therefore $\alpha \in \mathcal{J}K^t[G]$. \square

Again let $K^t[G]$ be given. If $x \in G$, we let $\mathbb{C}_G^t(x) = \{g \in G \mid \bar{g}\bar{x} = \bar{x}\bar{g}\}$ be its *twisted centralizer*. Then $\mathbb{C}_G^t(x)$ is clearly a subgroup of G contained in $\mathbb{C}_G(x)$. Furthermore, if $g \in \mathbb{C}_G^t(x)$, then $\bar{g}^{-1}\bar{x}\bar{g} = \lambda_x(g)\bar{x}$ and $\lambda_x: \mathbb{C}_G^t(x) \rightarrow K^\bullet$ is a linear character with kernel precisely equal to $\mathbb{C}_G^t(x)$. Note that if x has finite order n , then $\bar{g}^{-1}\bar{x}^n\bar{g} = \bar{x}^n$ and therefore $\lambda_x(g)^n = 1$. Thus, in this case, the image of λ_x is contained in the finite cyclic group $\{k \in K \mid k^n = 1\}$, and in particular, the index $|\mathbb{C}_G(x) : \mathbb{C}_G^t(x)|$ divides n and is finite. Recall that if $\alpha = \sum_x a_x \bar{x} \in K^t[G]$, then the *support* of α is the finite subset of G given by

$$\text{supp } \alpha = \{x \in G \mid a_x \neq 0\}.$$

In view of Proposition 2.2, we will have to deal with nilpotent ideals in $K^t[G]$, and the Delta methods are appropriate tools for such a task. Specifically, the *Delta methods* are a series of arguments, based on a simple coset counting procedure, which can handle linear identities in group algebras. For the most part, they are used to reduce infinite group problems to the simpler finite case, but by keeping track of the bounds involved, they can also yield information on finite groups. To start with, let

$$\Delta = \Delta(G) = \{x \in G \mid |G : \mathbb{C}_G(x)| < \infty\}$$

and, for each integer n , set

$$\Delta_n = \Delta_n(G) = \{ x \in G \mid |G : \mathbb{C}_G(x)| \leq n \}.$$

Then Δ is a characteristic subgroup of G , its *finite conjugate center*, and each Δ_n is a characteristic subset containing 1 and closed under inverses. Furthermore, $\Delta_n \Delta_m \subseteq \Delta_{nm}$ and, since G is locally finite, Δ is generated by the finite normal subgroups of G (see [12, Lemma 4.1.8]). Note that G is said to be an *f.c. group* if $G = \Delta(G)$.

The following sequence of lemmas is the twisted analog of the argument given in [13]. Since we are only concerned with locally finite groups, we have formulated these results, using the finite group H , to obtain what we need as quickly as possible. Presumably, these lemmas hold in somewhat greater generality.

Lemma 2.4. *Let H be a finite subgroup of G of order $|H| = h$, and let $\alpha, \beta \in K^t[G]$ satisfy*

$$\alpha \bar{x} \beta = 0 \quad \text{for all } x \in G.$$

Suppose that $|\text{supp } \beta| = b$ and that $\alpha = \alpha' + \alpha''$ with $\alpha', \alpha'' \in K^t[H]$ and with $\text{supp } \alpha' \subseteq \Delta_k(G)$ for some integer k . Then either $\alpha' \beta = 0$ or there exists an element $y \in \text{supp } \alpha''$ with

$$|G : \mathbb{C}_G(y)| \leq (hk)^h (hb).$$

Proof. Since $\alpha', \alpha'' \in K^t[H]$, it follows that both these elements have support size at most $|H| = h$. Suppose now that $\alpha' \beta \neq 0$ and take v to be a fixed group element in its support. Consider any pair of elements y, z with $y \in \text{supp } \alpha''$ and $z \in \text{supp } \beta$. If y is conjugate to vz^{-1} in G , then we can choose $g_{y,z} \in G$ with $g_{y,z}^{-1} y g_{y,z} = vz^{-1}$. Otherwise, we make no such choice. In this way, we obtain at most $|\text{supp } \alpha''| |\text{supp } \beta| \leq hb$ such elements $g_{y,z}$.

If $u \in \text{supp } \alpha' \subseteq \Delta_k \cap H$, then $|G : \mathbb{C}_G(u)| \leq k$ since $u \in \Delta_k$. Furthermore, $|\mathbb{C}_G(u) : \mathbb{C}_G^t(u)| \leq h$ since the order of u divides $|H| = h$. Thus $|G : \mathbb{C}_G^t(u)| \leq hk$ and, since $|\text{supp } \alpha'| \leq h$, it follows that

$$W = \bigcap_{u \in \text{supp } \alpha'} \mathbb{C}_G^t(u)$$

is a subgroup of G of finite index $|G : W| \leq (hk)^h$. Note that if $w \in W$, then \bar{w} centralizes all \bar{u} with $u \in \text{supp } \alpha'$ and therefore \bar{w} centralizes α' .

We show now that $W \subseteq \bigcup_{y,z} \mathbb{C}_G(y) g_{y,z}$, where the union is over all appropriate pairs y, z with $y \in \text{supp } \alpha''$ and $z \in \text{supp } \beta$. To this end, fix $w \in W$. Then, since \bar{w} centralizes α' , the hypothesis implies that

$$0 = \bar{w}^{-1} \alpha \bar{w} \beta = \bar{w}^{-1} (\alpha' + \alpha'') \bar{w} \beta = \alpha' \beta + \bar{w}^{-1} \alpha'' \bar{w} \beta.$$

Thus v belongs to the support of $\bar{w}^{-1}\alpha''\bar{w}\beta = -\alpha'\beta$, so there exist $y \in \text{supp } \alpha''$ and $z \in \text{supp } \beta$ with $w^{-1}ywz = v$. In other words, $w^{-1}yw = vz^{-1}$, so y and vz^{-1} are conjugate in G and hence

$$w^{-1}yw = vz^{-1} = g_{y,z}^{-1}yg_{y,z}.$$

Consequently, $w \in \mathbb{C}_G(y)g_{y,z}$ and this claim is proved.

Finally, since $|G : W| \leq (hk)^h$, we conclude that G is a union of right cosets of the subgroups $\mathbb{C}_G(y)$ with $y \in \text{supp } \alpha''$ and that at most $(hk)^h(hb)$ terms occur. Hence, by [12, Lemma 5.2.2], we conclude that some y satisfies

$$|G : \mathbb{C}_G(y)| \leq (hk)^h(hb)$$

and the result follows since $y \in \text{supp } \alpha''$. \square

The coset counting argument used above typifies the Delta method. Note that we did not assume that α' and α'' had disjoint supports. Now, for each integer n , we define $\theta_n : K^t[G] \rightarrow K^t[G]$ by

$$\theta_n \left(\sum_{x \in G} a_x \bar{x} \right) = \sum_{x \in \Delta_n} a_x \bar{x}.$$

In other words, θ_n is the K -linear map determined by $\theta_n(\bar{x}) = \bar{x}$ if $x \in \Delta_n(G)$ and $\theta_n(\bar{x}) = 0$ if $x \in G \setminus \Delta_n(G)$. Obviously θ_n is not an algebra homomorphism; it is just a K -linear operator on the twisted group algebra.

Lemma 2.5. *Let $K^t[G]$ be given and let H be a finite subgroup of G of order $|H| = h$. Suppose that $\alpha_1, \alpha_2, \dots, \alpha_m \in K^t[H]$ and that*

$$\alpha_1 \bar{x}_1 \alpha_2 \bar{x}_2 \cdots \alpha_{m-1} \bar{x}_{m-1} \alpha_m = 0 \quad \text{for all } x_1, x_2, \dots, x_{m-1} \in G.$$

Let k be a fixed integer and set $f(k) = f_m(k) = h^{m+h}k^{m(h+1)}$. Then either

$$\theta_k(\alpha_1)\theta_k(\alpha_2) \cdots \theta_k(\alpha_m) = 0$$

or there exists $y \in \text{supp } \alpha_i$, for some i , with $y \in \Delta_{f(k)} \setminus \Delta_k$.

Proof. We assume that no such $y \in \text{supp } \alpha_i$ exists with $y \in \Delta_{f(k)} \setminus \Delta_k$. For convenience, let $\alpha_{m+1} = 1$ and observe that

$$\alpha_1 \bar{x}_1 \alpha_2 \bar{x}_2 \cdots \alpha_m \bar{x}_m \alpha_{m+1} = 0 \quad \text{for all } x_1, x_2, \dots, x_m \in G.$$

We show by induction on $j = 1, 2, \dots, m+1$ that

$$\theta_k(\alpha_1) \cdots \theta_k(\alpha_{j-1}) \alpha_j \bar{x}_j \alpha_{j+1} \bar{x}_{j+1} \cdots \bar{x}_m \alpha_{m+1} = 0$$

for all $x_j, x_{j+1}, \dots, x_m \in G$. The case $j = 1$ is given.

Now assume that the result holds for some $j \leq m$, fix $x_{j+1}, x_{j+2}, \dots, x_m \in G$, and set

$$\alpha = \theta_k(\alpha_1) \cdots \theta_k(\alpha_{j-1})\alpha_j, \quad \beta = \alpha_{j+1}\bar{x}_{j+1} \cdots \bar{x}_m\alpha_{m+1}.$$

Then the inductive hypothesis yields $\alpha\bar{x}\beta = 0$ for all $x \in G$. Furthermore, if

$$\alpha' = \theta_k(\alpha_1) \cdots \theta_k(\alpha_{j-1})\theta_k(\alpha_j), \quad \alpha'' = \theta_k(\alpha_1) \cdots \theta_k(\alpha_{j-1})[\alpha_j - \theta_k(\alpha_j)],$$

then clearly $\alpha', \alpha'' \in K^t[H]$, $\alpha = \alpha' + \alpha''$, and $\text{supp } \alpha' \subseteq \Delta_{kj}$. Since

$$|\text{supp } \beta| \leq \prod_{i=j+1}^m |\text{supp } \alpha_i| \leq h^{m-j}$$

and $1 \leq j \leq m$, the preceding lemma implies that either $\alpha'\beta = 0$ or there exists $z \in \text{supp } \alpha''$ with

$$|G : \mathbb{C}_G(z)| \leq (hk^j)^h (hh^{m-j}) \leq (hk^m)^h h^m.$$

If $\alpha'\beta = 0$, then we obtain

$$\theta_k(\alpha_1) \cdots \theta_k(\alpha_{j-1})\theta_k(\alpha_j)\alpha_{j+1}\bar{x}_{j+1} \cdots \bar{x}_m\alpha_{m+1} = 0,$$

as required.

Suppose on the other hand that such an element $z \in \text{supp } \alpha''$ exists. Then, by definition of α'' , it follows that $z = y_1 y_2 \cdots y_{j-1} y$ where $y_i \in \text{supp } \theta_k(\alpha_i)$ and $y \in \text{supp}[\alpha_j - \theta_k(\alpha_j)]$. Note that the latter containment implies that $y \in \text{supp } \alpha_j$ and $y \notin \Delta_k$. Furthermore, since $y = y_{j-1}^{-1} \cdots y_2^{-1} y_1^{-1} z$ and since each $y_i \in \Delta_k$, we have

$$|G : \mathbb{C}_G(y)| \leq k^{j-1} \cdot (hk^m)^h h^m \leq h^{m+h} k^{m(h+1)} = f_m(k).$$

But, by assumption, no such $y \in \Delta_{f(k)} \setminus \Delta_k$ exists with $y \in \text{supp } \alpha_j$. Thus the second possibility never occurs and the induction step is proved.

Finally, when $j = m + 1$, the inductive result asserts that

$$\theta_k(\alpha_1)\theta_k(\alpha_2) \cdots \theta_k(\alpha_m)\alpha_{m+1} = 0$$

and, since $\alpha_{m+1} = 1$, the result follows. \square

We close this section by combining the above with a pigeon hole argument.

Lemma 2.6. *Let $K^t[G]$ be given and let H be a finite subgroup of G of order $|H| = h$. Suppose that $\alpha_1, \alpha_2, \dots, \alpha_m \in K^t[H]$ and that*

$$\alpha_1 \bar{x}_1 \alpha_2 \bar{x}_2 \cdots \alpha_{m-1} \bar{x}_{m-1} \alpha_m = 0 \quad \text{for all } x_1, x_2, \dots, x_{m-1} \in G.$$

Then there exists an integer k with

$$1 \leq k \leq h^{(m+1)^h (h+2)^h}$$

and with

$$\theta_k(\alpha_1) \bar{z}_1 \theta_k(\alpha_2) \bar{z}_2 \cdots \theta_k(\alpha_{m-1}) \bar{z}_{m-1} \theta_k(\alpha_m) = 0$$

for all $z_1, z_2, \dots, z_{m-1} \in H$.

Proof. Since the result is obvious when $|H| = 1$, we can assume that $h > 1$. Now, for $i = 0, 1, \dots, h+1$, define the integers k_i by

$$k_i = h^{(m+1)^i (h+2)^i},$$

and observe that $h = k_0 < k_1 < \cdots < k_{h+1}$. Thus the $h+1$ sets $\Delta_{k_{i+1}} \setminus \Delta_{k_i}$ with $i = 0, 1, \dots, h$ are disjoint and, since $|H| = h$, it follows that

$$(\Delta_{k_{j+1}} \setminus \Delta_{k_j}) \cap H = \emptyset$$

for some $0 \leq j \leq h$.

Let $k = k_j$ be as above and note that

$$k \leq k_h = h^{(m+1)^h (h+2)^h}.$$

Furthermore, $f_m(k) = h^{m+h} k^{m(h+1)} = h^n$, where

$$n = m + h + m(h+1)(m+1)^j (h+2)^j \leq (m+1)^{j+1} (h+2)^{j+1}.$$

Thus, $f_m(k) \leq k_{j+1}$, and it follows that

$$(\Delta_{f_m(k)} \setminus \Delta_k) \cap H = \emptyset.$$

Now let u_1, u_2, \dots, u_m be any fixed elements of H , and observe that

$$\alpha_1^{\bar{u}_1} \bar{x}_1 \alpha_2^{\bar{u}_2} \bar{x}_2 \cdots \alpha_{m-1}^{\bar{u}_{m-1}} \bar{x}_{m-1} \alpha_m^{\bar{u}_m} = 0$$

for all $x_1, x_2, \dots, x_{m-1} \in G$. Therefore, we can apply the preceding lemma to this situation with k as given and with each α_i replaced by $\alpha_i^{\bar{u}_i} = \bar{u}_i^{-1} \alpha_i \bar{u}_i \in K^t[H]$. Since $(\Delta_{f_m(k)} \setminus \Delta_k) \cap H = \emptyset$, we obtain

$$\theta_k(\alpha_1^{\bar{u}_1}) \theta_k(\alpha_2^{\bar{u}_2}) \cdots \theta_k(\alpha_m^{\bar{u}_m}) = 0,$$

an equation which holds for all $u_1, u_2, \dots, u_m \in H$. Furthermore, Δ_k is a normal subset of G , so $\theta_k(\alpha_i^{\bar{u}_i}) = \theta_k(\alpha_i)^{\bar{u}_i}$, and therefore

$$\theta_k(\alpha_1)^{\bar{u}_1} \theta_k(\alpha_2)^{\bar{u}_2} \cdots \theta_k(\alpha_m)^{\bar{u}_m} = 0.$$

Finally, let z_1, z_2, \dots, z_{m-1} be any elements of H , set $u_1 = 1$, and define $u_i = (z_1 z_2 \cdots z_{i-1})^{-1}$ for $i = 2, 3, \dots, m$. Then, for some scalar c , we have

$$\begin{aligned} & \theta_k(\alpha_1) \bar{z}_1 \theta_k(\alpha_2) \bar{z}_2 \cdots \theta_k(\alpha_{m-1}) \bar{z}_{m-1} \theta_k(\alpha_m) \\ &= c \theta_k(\alpha_1)^{\bar{u}_1} \theta_k(\alpha_2)^{\bar{u}_2} \cdots \theta_k(\alpha_m)^{\bar{u}_m} \bar{u}_m^{-1} = 0 \end{aligned}$$

and the lemma is proved. \square

As we will see, this has an extremely useful consequence.

§3. THE SUBNORMAL CLOSURE

A local version of the subnormal closure was used to good effect in [18]. We begin this section by reviewing its basic properties. To start with, let $H \subseteq X$ be finite groups. Since the set of subnormal subgroups of X is closed under intersection, it follows that there is a unique smallest subnormal subgroup S of X which contains H (see [22] or [25]). This is called the *subnormal closure* of H in X , and we denote it by $S = H^{[X]}$. If H^S is the normal closure of H in S , then $H \subseteq H^S \triangleleft S \triangleleft\triangleleft X$, so the minimal nature of S implies that $S = H^S$. In fact, S is characterized by the two properties (i) $H \subseteq S \triangleleft\triangleleft X$, and (ii) $S = H^S$. Note that, if \bar{X} is a homomorphic image of X , and if H and S are as above, then $\bar{H} \subseteq \bar{S} \triangleleft\triangleleft \bar{X}$ and $\bar{S} = \bar{H}^{\bar{S}}$. Thus \bar{S} is the subnormal closure of \bar{H} in \bar{X} . In general, subnormal closures do not exist for arbitrary subgroups of infinite groups.

Now if $H \subseteq X \subseteq Y$ are all finite, then $H \subseteq H^{[Y]} \cap X \triangleleft\triangleleft X$. Thus the minimal nature of $H^{[X]}$ implies that $H^{[X]} \subseteq H^{[Y]} \cap X \subseteq H^{[Y]}$. This inclusion allows us to define a *local subnormal closure* for finite subgroups of locally finite groups, and we use the same notation. Specifically, if H is a finite subgroup of G , then we set

$$H^{[G]} = \bigcup_X H^{[X]}$$

where the union is over all finite subgroups X of G containing H . Note that, if G is finite, then the inclusion $H^{[X]} \subseteq H^{[Y]}$ immediately implies that the two possible meanings for $H^{[G]}$ are the same.

At times, it is necessary to work in the context of a fixed set of generating subgroups for G . Recall that \mathcal{L} is said to be a *local system* for G if \mathcal{L} is a collection of finite subgroups with the property that every finite subgroup H of G is contained in some $L \in \mathcal{L}$. In particular, if $L_1, L_2 \in \mathcal{L}$, then there exists some $L \in \mathcal{L}$ with $\langle L_1, L_2 \rangle \subseteq L$, and this property, along with $G = \langle L \mid L \in \mathcal{L} \rangle$, clearly characterizes a local system. The following is [18, Lemma 2.2].

Lemma 3.1. *Let H be a finite subgroup of G , and set $S = H^{[G]}$.*

- i. *S is a subgroup of G containing H .*
- ii. *If \mathcal{L} is a local system for G , then $\{H^{[L]} \mid L \in \mathcal{L}, L \supseteq H\}$ is a local system for the subgroup S .*
- iii. *If $A \text{ lsn } S$, then $A \text{ lsn } G$.*
- iv. *$S = H^S$ is the normal closure of H in S , and $\mathbb{N}_G(H) \subseteq \mathbb{N}_G(S)$.*

Obviously, parts (ii) and (iii) above will allow us to reduce the proof of Theorem 1.4 to certain local subnormal closures. However, it is the condition $S = H^S$ in (iv) which is really crucial.

When studying twisted group algebras, it is frequently useful to assume that the field of coefficients is algebraically closed. Thus, in Sections 3–6, K will always denote an algebraically closed field of characteristic $p > 0$. One consequence of this is as follows.

Lemma 3.2. *Let P be a normal p -subgroup of G and let $K^t[G]$ be given.*

- i. *$K^t[P] \cong K[P]$ via a diagonal change of basis, and $\mathcal{J}K^t[P]$ is a characteristic ideal of $K^t[P]$ of codimension 1.*
- ii. *$I = \mathcal{J}K^t[P] \cdot K^t[G]$ is an ideal of $K^t[G]$ contained in $\mathcal{J}K^t[G]$, and $K^t[G]/I$ is naturally isomorphic to $K^t[G/P]$, a uniquely determined twisted group algebra of G/P . Furthermore, the homomorphism $K^t[G] \rightarrow K^t[G/P]$ maps the group \mathcal{G} of trivial units of $K^t[G]$ onto the groups \mathcal{G}/\mathcal{P} of trivial units of $K^t[G/P]$ and maps $\mathcal{J}K^t[G]$ onto $\mathcal{J}K^t[G/P]$.*

Proof. (i) If A is a finite subgroup of P , then A is a finite p -group and [12, Lemma 1.2.10] implies that $K^t[A] \cong K[A]$. Thus $\mathcal{J}K^t[A]$ is a maximal ideal of $K^t[A]$ of codimension 1, and consequently (1) the set of nilpotent elements of $K^t[A]$ is an ideal, and (2) every element of $K^t[A]$ is the sum of a scalar and a nilpotent element. Since P is locally finite, properties (1) and (2) clearly carry over to $K^t[P]$, and therefore $\mathcal{J}K^t[P]$ is a characteristic ideal of $K^t[P]$ of codimension 1. Finally, let $\lambda: K^t[P] \rightarrow K^t[P]/\mathcal{J}K^t[P] = K$ be the natural homomorphism and, for each $x \in P$, let $\tilde{x} = \lambda(\bar{x})^{-1}\bar{x}$. Then \tilde{x} is the unique scalar multiple of \bar{x} with λ -image equal to 1, so it is clear that $\tilde{x}\tilde{y} = \widetilde{xy}$ for all $x, y \in P$. In other words, $K^t[P] \cong K[P]$ via a diagonal change of basis.

(ii) Since $P \triangleleft G$ and $\mathcal{J}K^t[P]$ is characteristic in $K^t[P]$, it follows that $\mathcal{J}K^t[P] = \bar{x}^{-1}\mathcal{J}K^t[P]\bar{x}$ for all $x \in G$. Therefore

$$I = \mathcal{J}K^t[P] \cdot K^t[G] = K^t[G] \cdot \mathcal{J}K^t[P]$$

is an ideal of $K^t[G]$ which is contained in $\mathcal{J}K^t[G]$ by Lemma 2.3. Furthermore, since $K^t[P]/\mathcal{J}K^t[P] = K$, it follows that $K^t[G]/I$ has a K -basis consisting of the images of those \bar{x} coming from a fixed transversal for P in G . Thus $K^t[G]/I$ is naturally isomorphic to a twisted group algebra $K^t[G/P]$. \square

Recall that if $\alpha = \sum a_x \bar{x} \in K^t[G]$, then the support of α is the finite subset of G given by $\text{supp } \alpha = \{x \in G \mid a_x \neq 0\}$. In addition, we call $H = \langle \text{supp } \alpha \rangle$ the *supporting subgroup* of α . Clearly H is the smallest subgroup of G with $\alpha \in K^t[H]$ and, since G is locally finite, H is finite. We say that $\beta \in K^t[G]$ is a *truncation* of α if $\beta = \sum' a_x \bar{x}$, where \sum' indicates a partial sum of the terms in α . Thus $\text{supp } \beta \subseteq \text{supp } \alpha$, and the coefficients in α and in β agree on the smaller set. Of course, β is a *proper truncation* if $\beta \neq 0$ or α .

Observe that $\theta_k(\alpha)$ is a truncation of α for any integer k , and other truncations occur as follows. For any subgroup D of G there is a natural $K^t[D]$ -bimodule projection map $\pi_D: K^t[G] \rightarrow K^t[D]$ given by

$$\pi_D: \sum_{x \in G} a_x \bar{x} \mapsto \sum_{x \in D} a_x \bar{x}.$$

Thus π_D is the linear extension of the map $G \rightarrow D \cup \{0\}$ which is the identity on D and zero on $G \setminus D$. Clearly, $\pi_D(\alpha)$ is a truncation of α .

The next two results, generalizing [18, Lemma 2.4], yield a very powerful tool. The lemma merely fixes notation. The proposition contains the real content, combining the work of the previous section with simple properties of the subnormal closure in finite groups.

Lemma 3.3. *Let $K^t[G]$ be given with $\mathcal{J}K^t[G] \neq 0$. Then there exist $\alpha \in K^t[G]$ and finite subgroups $A \subseteq B$ of G satisfying*

- i. α is an element of minimal nonzero support size in $\mathcal{J}K^t[G]$, $1 \in \text{supp } \alpha$, and $A = \langle \text{supp } \alpha \rangle$.
- ii. If L is any subgroup of G containing B , then no proper truncation β of α is contained in $\mathcal{J}K^t[L]$.

Proof. Choose α to be a nonzero element of $\mathcal{J}K^t[G]$ of minimal support size. If $g \in \text{supp } \alpha$, then $\bar{g}^{-1}\alpha \in \mathcal{J}K^t[G]$, $1 \in \text{supp } \bar{g}^{-1}\alpha$, and $|\text{supp } \bar{g}^{-1}\alpha| = |\text{supp } \alpha|$. Thus, by replacing α by $\bar{g}^{-1}\alpha$ if necessary, we can assume that $1 \in \text{supp } \alpha$. Let $A = \langle \text{supp } \alpha \rangle$ be the supporting subgroup of α .

Suppose $\beta_1, \beta_2, \dots, \beta_r$ are the finitely many proper truncations of α . Then $0 < |\text{supp } \beta_i| < |\text{supp } \alpha|$, so the minimal nature of $|\text{supp } \alpha|$ implies that no β_i is contained in $\mathcal{J}K^t[G]$. In particular, the right ideals $\beta_i K^t[G]$ are not nil, so there exist elements $\gamma_i \in K^t[G]$ with $\beta_i \gamma_i$ not nilpotent. Since G is locally finite, we can choose a finite subgroup B of G containing A and the supports of all the γ_i . Consequently, if L is any subgroup of G containing B , then no β_i is contained in the nil ideal $\mathcal{J}K^t[L]$. \square

In the following, $\mathbb{O}_p(N)$ denotes the largest normal p -subgroup of N and $\mathbb{O}^{p'}(N)$ is the smallest normal subgroup of N with p' -quotient. As will be apparent, there is a more general version of part (iii) below in which we replace the assumption

that N/P is a p' -group with the weaker condition $\mathcal{J}K^t[N/P] = 0$, but this is all we require. For convenience, we define the function $n(h)$ by

$$n(h) = h^{h(h+1)^h(h+2)^h}.$$

Proposition 3.4. *Let $K^t[G]$ be a twisted group algebra of the locally finite group G over the field K of characteristic $p > 0$. Suppose $\mathcal{J}K^t[G] \neq 0$, and choose α, A , and B as in the preceding lemma. Furthermore, let L be any finite subgroup of G containing B , let $H = A^{[L]}$ be the subnormal closure of A in L , and set $n = n(|H|)$. Then we have*

- i. $H = \mathbb{O}^{p'}(H)$ is generated by p -elements.
- ii. If N is any subgroup of G normalized by H and satisfying $\mathcal{J}K^t[N] = 0$, then $H \subseteq \Delta_n(NH) \subseteq \Delta(NH)$.
- iii. If N is any subgroup of G normalized by H and if $N/\mathbb{O}_p(N)$ is a p' -group, then $HP/P \subseteq \Delta_n(NH/P) \subseteq \Delta(NH/P)$, where $P = \mathbb{O}_p(N)$.

Proof. By assumption, $\langle \text{supp } \alpha \rangle = A \subseteq H \triangleleft L$, and therefore Lemma 2.3 implies that $\mathcal{J}K^t[H] \subseteq \mathcal{J}K^t[L]$. In particular, since no proper truncation of α is contained in $\mathcal{J}K^t[L]$, the same must be true for $\mathcal{J}K^t[H]$. In other words, $\alpha \in \mathcal{J}K^t[H]$, by Lemma 2.1, but no proper truncation of α is in this radical. Now let $|H| = h$, define the integer q by

$$q = h^{(h+1)^h(h+2)^h},$$

and observe that $n = n(h) = q^h$.

(i) If $Q = \mathbb{O}^{p'}(H)$ is the normal subgroup of H generated by its p -elements, then $|H : Q|$ is prime to p . Thus, by Proposition 2.2, $\mathcal{J}K^t[H] = \mathcal{J}K^t[Q] \cdot K^t[H]$ and consequently $\pi_Q(\alpha) \in \mathcal{J}K^t[Q] \subseteq \mathcal{J}K^t[H]$. But $\pi_Q(\alpha)$ is a truncation of α and $\pi_Q(\alpha) \neq 0$ since $1 \in \text{supp } \alpha$. Hence $\pi_Q(\alpha) = \alpha$, and therefore $A = \langle \text{supp } \alpha \rangle \subseteq Q \triangleleft H$. In particular, since H is the subnormal closure of A in L and since $Q \triangleleft L$, we conclude that $H = Q$ is generated by p -elements.

(ii) Let N be a subgroup of G normalized by H and satisfying $\mathcal{J}K^t[N] = 0$, and set $X = NH$. Then N is a normal subgroup of X of finite index at most $|H| = h$, and therefore $(\mathcal{J}K^t[X])^h \subseteq \mathcal{J}K^t[N] \cdot K^t[X] = 0$ by Proposition 2.2. In particular, since $\alpha \in \mathcal{J}K^t[G] \cap K^t[X] \subseteq \mathcal{J}K^t[X]$ by Lemma 2.1, we have

$$\alpha \bar{x}_1 \alpha \bar{x}_2 \cdots \alpha \bar{x}_{h-1} \alpha = 0 \quad \text{for all } x_1, x_2, \dots, x_{h-1} \in X$$

and Lemma 2.6 applies with $m = h$. Therefore, by definition of q , there exists an integer $k \leq q$ with

$$\theta_k(\alpha) \bar{z}_1 \theta_k(\alpha) \bar{z}_2 \cdots \theta_k(\alpha) \bar{z}_{h-1} \theta_k(\alpha) = 0$$

for all $z_1, z_2, \dots, z_{h-1} \in H$. In other words, $\theta_k(\alpha)K^t[H]$ is a nilpotent right ideal of $K^t[H]$, and hence $\theta_k(\alpha) \in \mathcal{J}K^t[H]$. But $\theta_k(\alpha)$ is a nonzero truncation of α , so by properties of H and α we have $\theta_k(\alpha) = \alpha$ and hence $\text{supp } \alpha \subseteq \Delta_k(X) \subseteq \Delta_q(X)$.

Finally, let $T = \Delta_q(X) \cap H$, so that T is a normal subset of H containing $\text{supp } \alpha$. Since $\text{supp } \alpha$ generates A and since conjugates of A generate $H = A^H$, it follows that T generates H . Furthermore, since $1 \in T$, we have the ascending chain $1 = T^0 \subseteq T^1 \subseteq T^2 \cdots$ of subsets of H , and it is clear that once two adjacent terms of this sequence are equal, then the chain stabilizes. Consequently, since $|H| = h$, the chain must stabilize by the h th term. Therefore, since $n = q^h$, we have

$$H = T^h \subseteq \Delta_q(X)^h \subseteq \Delta_{q^h}(X) = \Delta_n(X) \subseteq \Delta(X),$$

as required.

(iii) Here we just indicate how the preceding argument should be modified. To start with, let N be any subgroup of G normalized by H such that $N/\mathbb{O}_p(N)$ is a p' -group. Set $X = NH$ and observe that both $P = \mathbb{O}_p(N)$ and N are normal in X and that $|X : N| \leq |H| = h$. In view of Lemma 3.2, let $\tilde{\cdot} : K^t[X] \rightarrow K^t[X/P]$ denote the natural homomorphism with kernel $I = \mathcal{J}K^t[P] \cdot K^t[X] \subseteq \mathcal{J}K^t[X]$. Then I is a nil ideal and $I \cap K^t[H] \subseteq \mathcal{J}K^t[H]$. Suppose first that $\alpha \in K^t[P]$. Then $A = \langle \text{supp } \alpha \rangle \subseteq P \cap H \triangleleft H$, so $H = A^H$ implies that $P \cap H = H$. In other words, $H \subseteq P$ and therefore $HP/P = \langle 1 \rangle$ is trivially contained in $\Delta_n(X/P)$. Thus, we can assume that $\alpha \notin K^t[P]$.

Now write $\alpha = \sum_{x \in S} \alpha_x \bar{x}$ where the elements of S are in distinct cosets of $P \cap H$ in H , where each α_x is a nonzero element of $K^t[P \cap H]$, and where $1 \in S$. Then, using \tilde{x} to denote the image of \bar{x} , we see that the image of α under $\tilde{\cdot}$ is given by $\tilde{\alpha} = \sum_{x \in S} \tilde{\alpha}_x \tilde{x}$ and each $\tilde{\alpha}_x$ is contained in K . If $\tilde{\alpha}_x = 0$ for some $x \in S$, then $\alpha_x \bar{x}$ is a nonzero truncation of α contained in $I \cap K^t[H] \subseteq \mathcal{J}K^t[H]$. Thus, by properties of α and H , we have $\alpha = \alpha_x \bar{x}$, so $x = 1$ and $\alpha \in K^t[P]$, contrary to our assumption. In other words, all $\tilde{\alpha}_x$ are nonzero, and this says that $\text{supp } \tilde{\alpha} = (\text{supp } \alpha)P/P$ and therefore $\langle \text{supp } \tilde{\alpha} \rangle = AP/P$. Of course $H = A^H$ implies that AP/P is contained in no proper normal subgroup of HP/P .

Next, $\alpha \in \mathcal{J}K^t[X]$ implies that $\tilde{\alpha} \in \mathcal{J}K^t[X/P]$. Furthermore, if $\tilde{\beta}$ is a proper truncation of $\tilde{\alpha}$, then $\tilde{\beta}$ is the image of a proper truncation β of α . Thus, since $\beta \notin \mathcal{J}K^t[H]$ and $\mathcal{J}K^t[H]/(I \cap K^t[H]) = \mathcal{J}K^t[H/(H \cap P)]$, it follows that $\tilde{\beta} \notin \mathcal{J}K^t[HP/P]$. Finally, since N/P is a p' -group, $K^t[N/P]$ is semiprimitive and hence we can apply the argument of part (ii) to conclude that $\text{supp } \tilde{\alpha} \subseteq \Delta_q(X/P)$ and therefore that $HP/P \subseteq \Delta_n(X/P) \subseteq \Delta(X/P)$. \square

Another useful tool is the twisted analog of [13, Lemma 3.7], and we sketch its proof below. Recall that if N is any subgroup of G , then its *almost* or *finitary centralizer* is defined by

$$\mathbb{D}_G(N) = \{ x \in G \mid |N : \mathbb{C}_N(x)| < \infty \}.$$

It follows easily that $\mathbb{D}_G(N)$ is a subgroup of G , and if $N \triangleleft G$, then $\mathbb{D}_G(N)$ is the normal subgroup of G consisting of those elements which act in a *finitary manner* on N . Note that

$$\mathbb{D}_G(G) = \Delta(G) = \{x \in G \mid |G : \mathbb{C}_G(x)| < \infty\}.$$

Proposition 3.5. *If $N \triangleleft G$ with $\mathcal{J}K^t[N]$ nilpotent, then*

$$\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G]$$

where $D = \mathbb{D}_G(N)$. In particular,

$$\pi_D(\mathcal{J}K^t[G]) = \mathcal{J}K^t[D] \subseteq \mathcal{J}K^t[G].$$

Proof. Fix $\alpha \in \mathcal{J}K^t[G]$ and let $\delta \in K^t[D]$ be arbitrary. Since G is locally finite, there exists a subgroup L of G containing N with L/N finite and with $\alpha, \delta \in K^t[L]$. By hypothesis, $\mathcal{J}K^t[N]$ is nilpotent and therefore, by Proposition 2.2, $\mathcal{J}K^t[L]$ is also nilpotent. Thus, using the known structure of the nilpotent radical of $K^t[L]$ in [14, Proposition 2.8], we obtain $\mathcal{J}K^t[L] = \mathcal{J}K^t[\Delta(L)] \cdot K^t[L]$. Now observe that $\alpha\delta \in \mathcal{J}K^t[G] \cap K^t[L] \subseteq \mathcal{J}K^t[L]$ and that $\Delta(L) = \mathbb{D}_L(N) = D \cap L$ since $|L/N| < \infty$. Thus

$$\pi_D(\alpha)\delta = \pi_D(\alpha\delta) = \pi_{D \cap L}(\alpha\delta) \in \mathcal{J}K^t[\Delta(L)] \subseteq \mathcal{J}K^t[L].$$

In other words, $\pi_D(\alpha)\delta$ is nilpotent for all such δ , so $\pi_D(\alpha)K^t[D]$ is a nil right ideal of $K^t[D]$ and $\pi_D(\alpha) \in \mathcal{J}K^t[D]$.

We have therefore shown that $\pi_D(\mathcal{J}K^t[G]) \subseteq \mathcal{J}K^t[D]$, and hence we have

$$\mathcal{J}K^t[G] \subseteq \pi_D(\mathcal{J}K^t[G]) \cdot K^t[G] \subseteq \mathcal{J}K^t[D] \cdot K^t[G].$$

Of course, the reverse inclusion follows from Lemma 2.3. \square

§4. LOCALLY p -SOLVABLE GROUPS

The goal of this section is to obtain the twisted analog of [13, Lemma 4.1]. In other words, we will prove Theorem 1.4 for the case of locally p -solvable groups. This is, of course, not the full extent of the work in [13], but it is all that we require at this point in the argument. Now, in order to use the conclusion of Proposition 3.4, we need some information on finitary actions of p -groups on locally finite p' -groups. The following sequence of three lemmas from [13] will be adequate for our purposes. To start with, we have [13, Lemma 2.8].

Lemma 4.1. *Let P be a finite p -group acting on a finite $\text{GF}(q)$ -vector space V for some prime $q \neq p$. Set $X_n = \{x \in P \mid |V : \mathbb{C}_V(x)| \leq n\}$ for some integer n , and suppose that P is generated by X_n . Then every nonprincipal irreducible P -submodule of V has size at most n^2 .*

Next, the argument of [13, Lemma 2.9] yields

Lemma 4.2. *Let P be a finite p -group acting faithfully on a locally finite p' -group Q , and let $X_n = \{x \in P \mid |Q : \mathbb{C}_Q(x)| \leq n\}$ for some integer n . If P is generated by X_n and if $x \in X_n$, then the commutator group $[P, x]$ has order at most $(n^2!)^{n^2}$.*

Finally, we offer a slight extension of [13, Lemma 2.9]. The second part, involving the group M , is needed in Section 7 to handle a particular subcase when $p = 2$.

Lemma 4.3. *Let P be a finite p -group acting faithfully as automorphisms on a locally finite p' -group Q .*

- i. *If $x \in P$ with $|Q : \mathbb{C}_Q(x)| = n$, then the commutator group $[[P, x], x]$ has order at most $(n^2!)^{n^2}$.*
- ii. *Suppose $P \triangleleft M$ and that M acts on Q . If x is a p -element of M with $|Q : \mathbb{C}_Q(x)| = n$, then the group $[[P, x]^M, x]$ has order at most $(n^4!)^{n^4}$.*

Proof. (i) Let $H = \langle x \rangle^P$ be the normal closure $\langle x \rangle$ in P . Since each such conjugate x^y of x satisfies $|Q : \mathbb{C}_Q(x^y)| = n$, we conclude from the previous lemma applied to H that $[[P, x], x] \subseteq [H, x]$ has order at most $(n^2!)^{n^2}$.

(ii) Since $[[P, x]^M, x]$ is contained in P , we can clearly assume that M acts faithfully on Q . Now note that if $y \in P$, then $[y, x] = y^{-1}x^{-1}yx = (x^{-1})^y x$, and therefore $|Q : \mathbb{C}_Q([y, x])| \leq n^2$. Furthermore, $[P, x] \subseteq P$, so $H = \langle [P, x]^M, x \rangle$ is a p -subgroup of M generated by elements z with $|Q : \mathbb{C}_Q(z)| \leq n^2$. The preceding lemma applied to H now implies that $[[P, x]^M, x] \subseteq [H, x]$ has order at most $(n^4!)^{n^4}$, as required. \square

Recall that $\mathbb{O}_p(G)$ is the largest normal p -subgroup of the locally finite group G and that $\mathbb{O}_{p'}(G)$ denotes the largest normal p' -subgroup of G . Furthermore, following [13], we define

$$\mathbb{T}_p(G) = \mathbb{O}_{pp'pp'}(G).$$

Thus $\mathbb{T}_p(G)$ is the top term of the sequence of normal subgroups of G given by

$$\langle 1 \rangle \subseteq \mathbb{O}_p(G) \subseteq \mathbb{O}_{pp'}(G) \subseteq \mathbb{O}_{pp'p}(G) \subseteq \mathbb{O}_{pp'pp'}(G) = \mathbb{T}_p(G)$$

where the quotients are alternately p -groups and p' -groups, and observe that

$$\mathbb{O}_{p'p}(G) \subseteq \mathbb{T}_p(G).$$

The following is essentially [13, Lemma 2.6].

Lemma 4.4. *Let G be a locally finite group with $\mathbb{O}_p(G) = \langle 1 \rangle = \mathbb{O}_{p'}(G)$. If L is any finite subgroup of G , then there exists a finite subgroup H of G with $L \subseteq H$ and $L \cap \mathbb{T}_p(H) = \langle 1 \rangle$.*

Next, we need a Hall-Higman result.

Lemma 4.5. *Let H be a finite p -solvable group which acts faithfully as a characteristic p linear group on the vector space V , and suppose that, for all p -elements $x \in H$, we have $[[V, x], x] = 0$. Then $H = \mathbb{O}_{pp'pp'}(H) = \mathbb{T}_p(H)$.*

Proof. If x is a p -element of H , then the commutator equation $[[V, x], x] = 0$ asserts that $(x - 1)^2 = 0$ on V . Hence, $0 = (x - 1)^p = x^p - 1$, and it follows that $x^p = 1$. In other words, if P is a Sylow p -subgroup of H , then P has period 1 or p .

We show now that $P/\mathbb{O}_p(H)$ is abelian. Since this is clear for $p = 2$, it suffices to assume that $p \geq 3$. Hence, in the notation of [5], all nonidentity p -elements of H are exceptional. Now let V_1, V_2, \dots, V_t be the composition factors of V as a H -module, and let L_i denote the kernel of H on V_i . Then $\bigcap_i L_i = \mathbb{O}_p(H)$ and $\mathbb{O}_p(H/L_i) = \langle 1 \rangle$. Since P has period p , it follows from the latter and [5, Theorem 2.1.2] that PL_i/L_i is abelian for each i . Thus $P' \subseteq \bigcap_i L_i = \mathbb{O}_p(H)$, and $P/\mathbb{O}_p(H)$ is indeed abelian.

Finally, consider $\bar{H} = H/\mathbb{O}_{pp'}(H)$. Then \bar{H} is p -solvable with $\mathbb{O}_{p'}(\bar{H}) = \langle 1 \rangle$ and hence, by [5, Lemma 1.2.3], $\mathbb{O}_p(\bar{H})$ contains its centralizer in \bar{H} . But, by the above, the Sylow p -subgroups of \bar{H} are abelian and therefore they centralize $\mathbb{O}_p(\bar{H})$. Thus $\mathbb{O}_p(\bar{H})$ is a Sylow p -subgroup of \bar{H} , so $\bar{H} = \mathbb{O}_{pp'}(\bar{H})$ and $H = \mathbb{O}_{pp'pp'}(H) = \mathbb{T}_p(H)$, as required. \square

As a consequence, we obtain the following rather technical, group theoretic lemma which turns out to have two distinct applications to the semiprimitivity problem.

Lemma 4.6. *Let G be a locally finite, locally p -solvable group with $\mathbb{O}_p(G) = \langle 1 \rangle = \mathbb{O}_{p'}(G)$, and let H be a finite subgroup of G . Suppose that, for every p -element $x \in H$ and for every finite p -subgroup R of G containing x , the commutator group $R_x = [[R, x], x]$ has order bounded by a fixed integer m . Then $H = \mathbb{T}_p(H)$.*

Proof. Let P be a Sylow p -subgroup of H . If R is any finite p -subgroup of G containing P , then the hypothesis implies that

$$f(R) = \sum_{x \in P} |R_x| \leq m \cdot |P|$$

where, as above, $R_x = [[R, x], x]$. Thus, we can choose an appropriate finite p -subgroup R with $f(R)$ maximal. Now $\langle R, H \rangle$ is a finite subgroup of G , so Lemma 4.4 implies that there exists a finite subgroup $W \supseteq \langle R, H \rangle$ with $\langle R, H \rangle \cap \mathbb{T}_p(W) = \langle 1 \rangle$. Furthermore, since $N = \mathbb{O}_{p'}(W) \subseteq \mathbb{T}_p(W)$ and W is p -solvable, [5, Lemma 1.2.5] implies that $\langle R, H \rangle$ acts faithfully on the Frattini quotient V of $N/\mathbb{O}_{p'}(N)$.

Now the p -group R normalizes N , so it follows that R must normalize a Sylow p -subgroup S of N . Then $SR = S \rtimes R$ is a finite p -subgroup of G containing P and, for any $x \in P$, we have

$$(SR)_x = [[SR, x], x] \supseteq [[S, x], x] \cdot [[R, x], x] = S_x \cdot R_x.$$

Thus the maximality of $f(R)$ implies that $[[S, x], x] = \langle 1 \rangle$ for all $x \in P$. But $N = S\mathbb{O}_{p'}(N)$ so, in additive notation, this yields $[[V, x], x] = 0$ for all such x .

Finally, we consider the action of H on V . To start with, $[[V, x], x] = 0$ for all $x \in P$, and hence this holds for all p -elements x of H . Next, H acts faithfully as a linear group on the $\text{GF}(p)$ -vector space V . Thus, since H is p -solvable, the previous lemma yields the result. \square

We can now combine the preceding ingredients to prove what is essentially the main result of this section.

Proposition 4.7. *Let G be a locally finite, locally p -solvable group and let $K^t[G]$ be a twisted group algebra of G over the algebraically closed field K of characteristic $p > 0$. If $\mathbb{O}_p(G) = \langle 1 \rangle = \mathbb{O}_{p'}(G)$, then $K^t[G]$ is semiprimitive.*

Proof. Suppose by way of contradiction that $\mathcal{J}K^t[G] \neq 0$, and let α, A , and B satisfy the conclusion of Lemma 3.3. Then, by Lemma 4.4, there exists a finite subgroup $L \supseteq B$ of G with $B \cap \mathbb{T}_p(L) = \langle 1 \rangle$, and we set $H = A^{[L]}$. Thus α, A, B, L , and H satisfy the conclusion of Proposition 3.4. In particular, H is generated by p -elements, and parts (ii) and (iii) of that proposition hold with $n = n(|H|)$.

Let x be any p -element of H and let R be any finite p -subgroup of G containing x . Since $\langle R, L \rangle$ is a finite subgroup of G , Lemma 4.4 implies that there exists another finite subgroup $W \supseteq \langle R, L \rangle$ with $\langle R, L \rangle \cap \mathbb{T}_p(W) = \langle 1 \rangle$. Observe that $N = \mathbb{O}_{pp'}(W) \subseteq \mathbb{T}_p(W)$, and that $Q = N/\mathbb{O}_p(N)$ is a p' -group. Thus, since H normalizes N , Proposition 3.4(iii) implies that $H\mathbb{O}_p(N)/\mathbb{O}_p(N) \subseteq \Delta_n(HN/\mathbb{O}_p(N))$, and hence $|Q : \mathbb{C}_Q(x)| \leq n$. Furthermore, since $R \cap N = \langle 1 \rangle$ and W is p -solvable, [5, Lemma 1.2.3] implies that R acts faithfully on $Q = \mathbb{O}_{pp'}(W)/\mathbb{O}_p(W)$. Thus, we conclude from Lemma 4.3(i) that $R_x = [[R, x], x]$ has order at most $(n^2!)^{n^2}$.

In other words, G and H satisfy the hypotheses of the preceding lemma with $m = (n^2!)^{n^2}$, and hence we conclude that $H = \mathbb{T}_p(H)$. Furthermore, since $H \triangleleft\triangleleft L$, it follows from [25, pages 246–247] that $H \subseteq \mathbb{T}_p(L)$. Thus $A \subseteq H \subseteq \mathbb{T}_p(L)$ and, since $B \cap \mathbb{T}_p(L) = \langle 1 \rangle$, we have $A = \langle 1 \rangle$ and $K^t[A] = K$. But

$$0 \neq \alpha \in \mathcal{J}K^t[G] \cap K^t[A] \subseteq \mathcal{J}K^t[A] = \mathcal{J}K = 0,$$

so this yields the required contradiction. \square

Next, we translate the above global assumption to a local one.

Corollary 4.8. *Let G be a locally finite, locally p -solvable group and let $K^t[G]$ be a twisted group algebra of G over the algebraically closed field K of characteristic $p > 0$. If G has no nonidentity locally subnormal subgroup, then $K^t[G]$ is semiprimitive.*

Proof. Since $\mathbb{O}_p(G) \triangleleft G$ is generated by locally subnormal subgroups, it follows from the hypothesis that $\mathbb{O}_p(G) = \langle 1 \rangle$ and hence that $\mathbb{O}_p(N) = \langle 1 \rangle$ for all $N \triangleleft G$. Now let $H = \mathbb{O}_{p'}(G)$ and observe that $K^t[H]$ is semiprimitive. Thus, by Proposition 3.5, $\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G]$ where $D = \mathbb{D}_G(H)$. Finally, we note that $\mathbb{O}_p(D) = \langle 1 \rangle$ and that

$$\mathbb{O}_{p'}(D) \subseteq D \cap \mathbb{O}_{p'}(G) = D \cap H = \Delta(H) = \langle 1 \rangle$$

since $\Delta(H)$ is generated by finite normal subgroups of $H \triangleleft G$. Thus the previous proposition implies that $\mathcal{J}K^t[D] = 0$ and hence $\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G] = 0$, as required. \square

The preceding two results hold without the algebraically closed assumption, but such considerations will be dealt with later on. We close this section by combining the previous corollary with earlier work on infinite simple groups. Note that, if A, B lsn G , then basic properties of subnormal subgroups of finite groups imply that both $A \cap B$ and $\langle A, B \rangle$ are also locally subnormal in G . Furthermore, if H is any subgroup of G , then $(A \cap H)$ lsn H . Consequently, if G is generated by its locally subnormal subgroups, then the same is true of any subgroup $H \subseteq G$.

Proposition 4.9. *Let $K^t[G]$ be a twisted group algebra of the locally finite group G over an algebraically field K of characteristic $p > 0$. Suppose that G has a finite subnormal series*

$$G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient G_{i+1}/G_i either

- i. locally p -solvable, or
- ii. an infinite simple group, or
- iii. generated by its locally subnormal subgroups.

If $\mathcal{J}K^t[G_0] = 0$ and if G has no nonidentity locally subnormal subgroup, then $K^t[G]$ is semiprimitive.

Proof. We proceed by induction on n , the case $n = 0$ being given. Suppose the result holds for $n - 1$ and set $H = G_{n-1} \triangleleft G$. Since H has no nonidentity locally subnormal subgroups, the inductive assumption implies that $\mathcal{J}K^t[H] = 0$. Thus, by Proposition 3.5, $\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G]$, where $D = \mathbb{D}_G(H)$. Now $D \cap H = \Delta(H)$ is generated by finite normal subgroups of $H \triangleleft G$. Thus, since G has no nonidentity locally subnormal subgroup, it follows that $D \cap H = \langle 1 \rangle$. In other words, D is isomorphic to a normal subgroup of $G/H = G_n/G_{n-1}$.

In view of the above, it suffices to show that $\mathcal{J}K^t[D] = 0$ and, since $D \triangleleft G$, we know at least that D has no nonidentity locally subnormal subgroup. We consider

the three cases in turn. First, if G/H is locally p -solvable, then the same is true of D , and Corollary 4.8 yields the result. Next, if G/H is infinite simple, then either $D \cong G/H$ or $D = \langle 1 \rangle$, and the main results of [15] and [19] imply that $K^t[D]$ is semiprimitive. Finally, if G/H is generated by locally subnormal subgroups, then the same is true of D . Thus $D = \langle 1 \rangle$ and the proposition is proved. \square

§5. THE p -SOLVABLE RADICAL

As we indicated in the Introduction, much of paper [18] carries over directly to the context of twisted group algebras. Here, we quickly review sections 3 and 4 of that work but, for later considerations, we use the *p -solvable radical p -sol S* rather than the *solvable radical sol S* .

Recall that a finite group S is said to be *semisimple* if it is a direct product of nonabelian simple groups. Here we revert to the classical notation of [22] and [25], rather than that of [2]. As is well known, if $S = M_1 \times M_2 \times \cdots \times M_k$, with each M_i nonabelian and simple, then every normal subgroup N of S is a partial direct product of the M_i 's. Thus N is also semisimple, and it follows easily by induction that every subnormal subgroup of S is normal and hence of the above form. For convenience, we call k the *width* of S , and we write $k = \text{wd } S$.

Next, if S is any finite group, then we let p -sol S denote its unique largest normal p -solvable subgroup. Thus, p -sol S is characteristic in S and, by [25, page 246], it contains all p -solvable subnormal subgroups of S . In particular, if $N \triangleleft\triangleleft S$, then it follows that p -sol $N = N \cap p$ -sol S . Of course, p -sol S contains sol S , the unique largest normal solvable subgroup of S .

Finally, we say that S is *p -solvable-by-semisimple* if S/p -sol S is semisimple. In this case, we define the *width* of S to be that of S/p -sol S , so that $\text{wd } S$ is the number of simple factors of the semisimple group S/p -sol S . Since p -sol $(S/p$ -sol $S) = \langle 1 \rangle$, it follows that no simple factor of S/p -sol S can be a p' -group. Thus, the simple factors of S/p -sol S correspond to the composition factors of S which are nonabelian simple and have order divisible by p . Now note that if $N \triangleleft\triangleleft S$, then

$$N/p$$
-sol $N = N/(N \cap p$ -sol $S) \cong N(p$ -sol $S)/p$ -sol $S \triangleleft\triangleleft S/p$ -sol S .

Thus N/p -sol N is semisimple, N is p -solvable-by-semisimple, and $\text{wd } N \leq \text{wd } S$. Furthermore, if $N \triangleleft S$, then $N(p$ -sol $S)/N$ is a normal p -solvable subgroup of S/N with

$$\frac{S/N}{N(p$$
-sol $S)/N} \cong \frac{S}{N(p$ -sol $S)} \cong \frac{S/p$ -sol $S}{N(p$ -sol $S)/p$ -sol $S}$.

Thus S/N is also p -solvable-by-semisimple and $\text{wd } N + \text{wd } S/N = \text{wd } S$.

Our first goal is to consider countable *locally p -solvable-by-semisimple groups*. In other words, we assume that $G = \bigcup_{i=1}^{\infty} L_i$ is an ascending union of the finite p -solvable-by-semisimple groups L_i , and we start with the twisted analog of [18, Lemma 3.4]. In view of Proposition 4.9, its proof carries over without difficulty.

Lemma 5.1. *Let $G = \bigcup_{i=1}^{\infty} L_i$ be an ascending union of the finite p -solvable-by-semisimple groups L_i and assume that there is an integer k such that each L_i has width $\leq k$. Then G has a finite subnormal series*

$$\langle 1 \rangle = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = G$$

such that each factor N_{i+1}/N_i is either a simple group or locally p -solvable. In particular, if G has no nonidentity locally subnormal subgroup, then $\mathcal{JK}^t[G] = 0$.

Next, we drop the assumption on the widths of the various L_i 's, and we obtain the twisted analog of [18, Lemma 3.5]. The proof here uses Proposition 3.4(ii) and the fact that the p -insulator condition, in [19, Lemma 7.4(ii)], applies equally well to twisted group algebras.

Lemma 5.2. *Let $G = \bigcup_{i=1}^{\infty} L_i$ be the ascending union of the finite p -solvable-by-semisimple subgroups L_i . If G has no nonidentity locally subnormal subgroup, then $K^t[G]$ is semiprimitive.*

Thus the mainly group theoretic arguments of [18, §3] transfer directly to the case of twisted group algebras and, for the same reason, so do the methods of [18, §4]. To see this, recall that a finite group H is said to be *reduced* if $\text{sol } H = \langle 1 \rangle$ or equivalently if $\mathbb{O}_q(H) = \langle 1 \rangle$ for all primes q . Similarly, we say that H is *p -reduced* if $p\text{-sol } H = \langle 1 \rangle$, and obviously any p -reduced group is necessarily reduced.

Now let H be an arbitrary finite group and let $\text{soc } H$ be the characteristic subgroup of H generated by its minimal normal subgroups. Since any two distinct minimal normal subgroups commute, it follows that the *socle* is in fact the direct product of certain of these subgroups. Furthermore, any minimal normal subgroup of H is either an elementary abelian q -group for some prime q or it is semisimple. In particular, if H is reduced, then $\text{soc } H$ is semisimple. The following basic properties are from [18, Lemma 4.2]. Note that part (iv) requires CFSG and a precise version of the Schreier Conjecture.

Lemma 5.3. *Let H be a finite reduced group.*

- i. *Any subnormal subgroup of H is also reduced.*
- ii. *The socle of H is semisimple and $\mathbb{C}_H(\text{soc } H) = \langle 1 \rangle$.*
- iii. *Write $\text{soc } H = M_1 \times M_2 \times \cdots \times M_k$ as a direct product of nonabelian simple groups. Then every automorphism of H stabilizes $\text{soc } H$ and permutes the factors M_1, M_2, \dots, M_k .*
- iv. *If we let*

$$N = \{ h \in H \mid h \text{ stabilizes all } M_i \} = \bigcap_i \mathbb{N}_H(M_i),$$

then $N^{(4)} = \text{soc } H$, where $N^{(4)}$ is the fourth derived subgroup of N .

- v. *The M_i 's above are precisely the minimal subnormal subgroups of H .*

Of course, if H is any finite group, then $\bar{H} = H/p\text{-sol } H$ is reduced. Thus the above lemma applies to \bar{H} and, in particular, we know that $\text{soc } \bar{H}$ is semisimple. We will be concerned with the inverse image of $\text{soc } \bar{H}$ in H and, for want of a better name, we call it the p -radical of H . Thus $S = p\text{-rad } H$ is a characteristic subgroup of H containing $p\text{-sol } H$ and satisfying $S/p\text{-sol } H = \text{soc}(H/p\text{-sol } H)$. Obviously, $p\text{-sol } S = p\text{-sol } H$ and S is p -solvable-by-semisimple.

Now suppose that $G = \bigcup_{i=1}^{\infty} L_i$ is an ascending union of the finite groups L_i . Then each $S_i = p\text{-rad } L_i$ is p -solvable-by-semisimple and part (iii) of the previous lemma implies that L_i permutes the finite set Ω_i of simple factors of $S_i/p\text{-sol } L_i$ by conjugation. Following the approach used in [3] and [4], this local structure can be combined to yield a permutation representation for G itself.

To start with, let \mathcal{F} be an ultrafilter on the natural numbers \mathcal{N} which contains all cofinite sets. Then all members of \mathcal{F} are infinite and there is a natural permutation action of the ultraproduct group $L = \prod_{\mathcal{F}} L_i$ on the set $\Omega = \prod_{\mathcal{F}} \Omega_i$. Furthermore, there is a natural homomorphism $\phi: G \rightarrow L$ given by $\phi(g) = \prod_{\mathcal{F}} \phi_i(g)$, where $\phi_i(g) = g$ if $g \in L_i$ and $\phi_i(g) = 1$ otherwise (see for example [9, page 66]). In particular, the combined homomorphism $G \rightarrow L \rightarrow \text{Sym}_{\Omega}$ yields a permutation representation of G on Ω . Of course, this action need not be faithful, but we do have some control over the kernel. Indeed, the following is essentially [18, Lemma 4.3].

Lemma 5.4. *With the above notation, if N is the kernel of the action of G on Ω , then the fourth derived subgroup $N^{(4)}$ is locally p -solvable-by-semisimple.*

We close this section with yet another semiprimivity result, namely the twisted analog of [18, Lemma 4.4]. Its proof, which we sketch below, makes fundamental use of the classification of primitive finitary permutation groups. Specifically, [23, Satz 9.4] (or see [24, page 228]) asserts that the only infinite groups of this nature are the finitary symmetric or alternating groups of various cardinalities. Note that neither the hypothesis nor the conclusion of Lemma 5.5 mentions ultraproducts.

Lemma 5.5. *Let $G = \bigcup_{i=1}^{\infty} L_i$ be an ascending union of the finite groups L_i , and let H be a subgroup of L_1 with $G = H^G$. Suppose that k is a fixed positive integer and that, for each i , H moves at most k points in its conjugation action on the set Ω_i of simple factors of $p\text{-rad } L_i/p\text{-sol } L_i$. Then G has a finite subnormal series*

$$N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = G$$

where N_0 is locally p -solvable-by-semisimple and where each factor N_{i+1}/N_i is either an f.c. group or a finitary alternating group. In particular, if G has no nonidentity locally subnormal subgroup, then $K^t[G]$ is semiprimitive.

Proof. We use the preceding ultraproduct notation. Specifically, \mathcal{F} is an ultrafilter on \mathcal{N} containing all cofinite subsets, and G has a permutation representation $\tilde{\cdot}: G \rightarrow \text{Sym}_{\Omega}$ on the set $\Omega = \prod_{\mathcal{F}} \Omega_i$. Furthermore, if N is the kernel of this representation, then we know from Lemma 5.4 that $N_0 = N^{(4)}$ is locally p -solvable-by-semisimple.

As in the proof of [18, Lemma 4.4], it follows that H moves at most k points of Ω , and thus $\tilde{H} \subseteq \text{FSym}_\Omega$. In particular, since $\tilde{G} \subseteq \text{Sym}_\Omega$ and $\tilde{G} = \tilde{H}^{\tilde{G}}$, [18, Proposition 4.1] implies that $\tilde{G} = G/N$ has a finite subnormal series with factors which are either f.c. groups or isomorphic to a suitable FAlt_∞ . Thus the same is clearly true for the group $G/N_0 = G/N^{(4)}$.

Finally, suppose G has no nonidentity locally subnormal subgroup. Then the same is true of N_0 , and Lemma 5.2 implies that $\mathcal{J}K^t[N_0] = 0$. With this observation, Proposition 4.9 now yields the result. \square

§6. FINITE WREATH PRODUCTS

As we will see, Lemma 5.5 handles the semiprimitivity problem when a certain finite subgroup H of G moves a bounded number of points in its action on the simple factors of the various p -rad L_i/p -sol L_i . Thus it remains to consider the case where the number of moved points is unbounded, and that is the goal of this section. The argument we use requires a close look at the irreducible projective representations of certain finite groups having structures similar to that of a wreath product. Indeed, all groups considered in this section are finite.

If C denotes the field of complex numbers, then a finite group G is said to be of *central type* if some twisted group algebra $C^t[G]$ is simple. As was shown in [6], any group of central type must be solvable, and in particular if G is nonabelian simple, then $C^t[G]$ must have at least two distinct irreducible representations. The latter observation can be verified directly from CFSG and we do this below in the more difficult modular case.

Let $K^t[G]$ be given with K an algebraically closed field of characteristic $p > 0$ and with G a finite group. If $x \in G$, then the conjugacy class of x is said to be *special* if $\mathbb{C}_G^t(x) = \mathbb{C}_G(x)$, and a fundamental result (see [8, Theorem 3.6.7(ii)]) asserts that the number of irreducible representations of $K^t[G]$ is precisely equal to the number of p -regular special classes of G . Since $\{1\}$ is clearly a p -regular special class, we can show that $K^t[G]$ has at least two distinct irreducible representations by exhibiting just one nonidentity class of this nature.

Now let $\mathbf{M}(G)$ denote the Schur multiplier of G and recall that $\mathcal{G} = K^\bullet \bar{G}$, the group of trivial units of $K^t[G]$, satisfies $\mathcal{G}/K^\bullet \cong G$ and hence it is a central extension of G . Thus, since $G = G'$ when G is a nonabelian simple group, basic properties of the Schur multiplier (see [8, Proposition 4.1.7]) imply that $\mathcal{G}' \cap K^\bullet$ is a homomorphic image of $\mathbf{M}(G)$. In particular, since

$$\mathbb{C}_G(x)/\mathbb{C}_G^t(x) \cong \{ \bar{x}^{-1} \bar{g}^{-1} \bar{x} \bar{g} \mid g \in \mathbb{C}_G(x) \} \subseteq \mathcal{G}' \cap K^\bullet,$$

it follows that $\mathbb{C}_G(x)/\mathbb{C}_G^t(x)$ is involved in $\mathbf{M}(G)$. In other words, it is isomorphic to a subgroup of a factor group of $\mathbf{M}(G)$. On the other hand, we know that $\mathbb{C}_G(x)/\mathbb{C}_G^t(x)$ is cyclic of order dividing the order of x , and therefore, if $x \in G$ has order prime to the order of $\mathbf{M}(G)$, then the class of x must be special. As a

consequence, suppose $|G|$ is divisible by two distinct primes r_1 and r_2 which do not divide $|\mathbf{M}(G)|$. Then G has elements x_i of order r_i and the classes of x_1 and x_2 are both special. Furthermore, since x_1 and x_2 have relatively prime orders, one of these elements is p -regular, and therefore G has a nonidentity p -regular special class. This observation will handle all but one of the groups below.

Lemma 6.1. *Let G be a finite nonabelian simple group. Then any twisted group algebra $K^t[G]$ has at least two distinct irreducible representations.*

Proof. We will use the table of Schur multipliers given in [2, pages 302–303] and the table of group order formulas given in [2, pages 135–136]. We start with the two families of simple groups having unbounded multipliers. Suppose first that $G = A_n(q)$ with $n \geq 4$. Then $|\mathbf{M}(G)|$ divides $q - 1$, and $(q^3 - 1)(q^4 - 1)$ divides $|G|$. Thus, the Miraculous Prime Theorem ([1, Corollary 1]) guarantees that $|G|$ is divisible by two distinct primes which do not divide $|\mathbf{M}(G)|$. Similarly, suppose $G = {}^2A_n(q)$ with $n \geq 5$ and $(n, q) \neq (5, 2)$. Then $|\mathbf{M}(G)|$ divides $q + 1$, and $(q^3 + 1)(q^5 + 1)$ divides $|G|$, so the Miraculous Prime Theorem again yields the result. The remaining simple groups all have fairly small Schur multipliers.

Suppose next that $|\mathbf{M}(G)|$ is divisible by at most one prime. Then, since $|G|$ is divisible by at least three primes, we can certainly find the required primes r_1 and r_2 . In particular, in view of the table in [2, pages 302–303] and the result of the preceding paragraph, we are left with the following eleven possibilities: $A_1(9)$, $A_2(4)$, $B_3(3)$, ${}^2A_3(3)$, ${}^2A_5(2)$, ${}^2E_6(2)$, Alt_6 , Alt_7 , $M(22)$, M_{22} , and Suz . Furthermore, each of these groups satisfies $|\mathbf{M}(G)| = 2^a 3^b$ for suitable a and b , and almost all of these groups have order divisible by at least four primes. Indeed, it follows from [2, pages 135–136] that we need only consider $A_1(9) \cong \text{Alt}_6$.

Finally, let $G = \text{Alt}_6$. Then $|G| = 2^3 3^2 5$ and $|\mathbf{M}(G)| = 6$, so the elements of order 5 yield p -regular special classes unless $p = 5$. On the other hand, if $p = 5$ then we use the fact that if $x = (1\ 2\ 3\ 4)(5\ 6) \in G$, then x has order 4 and $\mathbb{C}_G(x) = \langle x \rangle$. Thus \bar{x} is central in $K^t[\mathbb{C}_G(x)]$, so $\mathbb{C}_G^t(x) = \mathbb{C}_G(x)$ and hence, in this case, the class of x is special and p -regular. \square

Next we move on to the twisted group algebras of semisimple groups. As usual, S' denotes the commutator subgroup of S .

Lemma 6.2. *Let $K^t[S]$ be given with S a finite group satisfying $S' = S$.*

- i. *If $S = M_1 \times M_2 \times \cdots \times M_r$, then*

$$K^t[S] = K^t[M_1] \otimes K^t[M_2] \otimes \cdots \otimes K^t[M_r].$$

- ii. *If $Z \subseteq \mathbb{Z}(S)$ is a central subgroup of S , then $K^t[Z]$ is central in $K^t[S]$. Furthermore, if e is an idempotent of $K^t[Z]$ with $eK^t[Z] = eK \cong K$, then $eK^t[S] = K^t[S/Z]$, where the latter is a twisted group algebra of S/Z .*

Proof. Recall that if $x \in S$, then $\mathbb{C}_S(x)/\mathbb{C}_S^t(x)$ is cyclic and hence $\mathbb{C}_S(x)' \subseteq \mathbb{C}_S^t(x)$.

(i) Now let $x \in M_i$. If $j \neq i$, then $M_j \subseteq \mathbb{C}_S(x)$ and hence $M_j = M_j' \subseteq \mathbb{C}_S^t(x)$. In other words, $K^t[M_i]$ and $K^t[M_j]$ commute elementwise, and therefore dimension considerations yield the result.

(ii) Here we note that if $x \in Z$, then $S = \mathbb{C}_S(x)$ and thus $S = S' \subseteq \mathbb{C}_S^t(x)$. It follows that $K^t[Z]$ is central in $K^t[S]$ and, in particular, if e is as above, then e is a central idempotent of $K^t[S]$. Furthermore, if Y is a transversal for Z in S , then $K^t[S] = \sum_{y \in Y} K^t[Z]\bar{y}$ and hence $eK^t[Z] = \sum_{y \in Y} eK\bar{y} = K^{t'}[S/Z]$. \square

With this observation, we can obtain the following extension of [18, Lemma 5.1].

Lemma 6.3. *Let $G = S \rtimes H$ be a finite semidirect product and let $K^t[G]$ be given. Let $Z = \mathbb{Z}(S)$ be a p' -group, set $\tilde{S} = S/Z$, and suppose that*

- i. $\tilde{S} = \tilde{S}' = \tilde{S}_0 \times \tilde{S}_1 \times \cdots \times \tilde{S}_r$ is a direct product of $r + 1$ groups,
- ii. H stabilizes \tilde{S}_0 and permutes the remainings \tilde{S}_i 's,
- iii. H acts faithfully as a permutation group on the set $\{\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_r\}$, and
- iv. for $i > 0$, every twisted group algebra of \tilde{S}_i over K has at least $|H|$ distinct irreducible representations.

Then $\mathcal{J}K^t[G] \cap K^t[H] = 0$.

Proof. Since $\tilde{S} = \tilde{S}'$, we have $S = S'Z$ and hence $Z \cap S' = \mathbb{Z}(S')$, $S'/\mathbb{Z}(S') = \tilde{S}$, and $S' = S''$. In other words, S' has the same properties as S and, in addition, it is equal to its own commutator subgroup. Furthermore, since $\mathcal{J}K^t[G] \cap K^t[S' \rtimes H] \subseteq \mathcal{J}K^t[S' \rtimes H]$, it clearly suffices to prove that $\mathcal{J}K^t[S' \rtimes H] \cap K^t[H] = 0$. Therefore, by replacing S by S' if necessary, we can assume that $S = S'$.

Now according to Lemma 6.2(ii), $K^t[Z]$ is central in $K^t[S]$. In particular, since Z is a p' -group, $K^t[Z]$ is a commutative semisimple algebra and hence isomorphic to a finite direct sum of copies of the algebraically closed field K . It follows that if e is a (centrally) primitive idempotent of $K^t[Z]$, then $eK^t[Z] = eK \cong K$ and hence Lemma 6.2 yields

$$\begin{aligned} eK^t[S] &= K^{t'}[S/Z] = K^{t'}[\tilde{S}_0] \otimes K^{t'}[\tilde{S}_1] \otimes \cdots \otimes K^{t'}[\tilde{S}_r] \\ &= eK^t[S_0] \otimes eK^t[S_1] \otimes \cdots \otimes eK^t[S_r] \end{aligned}$$

where $S_i/Z = \tilde{S}_i$.

Since H normalizes Z , it permutes the (centrally) primitive idempotents of $K^t[Z]$ via conjugation by \bar{H} . In particular, if

$$W = H_e = \{h \in H \mid \bar{h}^{-1}e\bar{h} = e\}$$

is the stabilizer of e in H , then W is a subgroup of H which acts as automorphisms on $eK^t[S]$. Furthermore, since W stabilizes S_0 and permutes the remaining S_i 's, it follows that W permutes the tensor factors $eK^t[S_i] = K^{t'}[\tilde{S}_i]$ of $K^{t'}[\tilde{S}]$.

At this point, it is convenient to relabel the S_i 's and their quotients $\tilde{S}_i = S_i/Z$ using double subscripts so that $\{S_{i,1}, S_{i,2}, \dots, S_{i,r_i}\}$ is the i th orbit of the action of W on $\{S_1, S_2, \dots, S_r\}$. In particular, if W_i denotes the stabilizer (normalizer) of $S_{i,1}$ in W , then $|W : W_i| = r_i$ and we can choose right coset representatives $\{w_{i,1}, w_{i,2}, \dots, w_{i,r_i}\}$ so that $S_{i,j} = S_{i,1}^{w_{i,j}}$.

Now W_i normalizes $S_{i,1}$ and therefore it permutes the irreducible representations of $eK^t[S_{i,1}] = K^t[\tilde{S}_{i,1}]$ via conjugation by \bar{W}_i . Furthermore, by assumption (iv), there are at least $|W|$ of the latter, so there are at least $|W|/|W_i| = r_i$ orbits of these representations. In particular, we can let $\phi_{i,1}, \phi_{i,2}, \dots, \phi_{i,r_i}$ be irreducible representations of $eK^t[S_{i,1}]$ in distinct W_i -orbits, and we define the irreducible representation $\psi_{i,j}$ of $eK^t[S_{i,j}]$ by $\psi_{i,j} = \phi_{i,j}^{w_{i,j}}$. For convenience, let ψ_0 denote any irreducible representation of $eK^t[S_0]$.

Since K is algebraically closed, it now follows that

$$\psi = \psi_0 \otimes \left(\bigotimes_{i,j} \psi_{i,j} \right)$$

is an irreducible representation for

$$eK^t[S] = eK^t[S_0] \otimes \left(\bigotimes_{i,j} eK^t[S_{i,j}] \right)$$

and hence for $K^t[S] = eK^t[S] \oplus (1 - e)K^t[S]$. Of course, W normalizes S , so W permutes the irreducible representations of $K^t[S]$, and we claim that it acts regularly on the W -orbit of ψ .

Indeed, suppose $w \in W$ stabilizes ψ and let i, i', j, k be subscripts with $S_{i,j}^w = S_{i',k}$. Then $S_{i,j}$ and $S_{i',k}$ are in the same W -orbit, so clearly $i' = i$. Furthermore, since $\psi^w = \psi$, uniqueness of tensor factors implies that $\psi_{i,j}^w = \psi_{i,k}$. Thus,

$$\phi_{i,j}^{w_{i,j} w} = \psi_{i,j}^w = \psi_{i,k} = \phi_{i,k}^{w_{i,k}}$$

or equivalently $\phi_{i,j}^{w_{i,j} w w_{i,k}^{-1}} = \phi_{i,k}$. But $w_{i,j} w w_{i,k}^{-1} \in W_i$, so $\phi_{i,j}$ and $\phi_{i,k}$ are in the same W_i -orbit and therefore $j = k$. In other words, we have shown that w acts trivially as a permutation on $\{S_{i,j}\}$, so hypothesis (iii) implies that $w = 1$ and hence W acts regularly on the W -orbit of ψ . In fact, since $W = H_e$, it is now clear that H acts regularly on the H -orbit of ψ .

Finally, since $G = SH = S \rtimes H$, it follows from the above regularity that the induced representation

$$\psi^G = \psi \otimes_{K^t[S]} K^t[G] = \psi \otimes_K K^t[H]$$

is irreducible and hence $\psi^G(\mathcal{J}K^t[G]) = 0$. On the other hand, the restriction $(\psi^G)_H$ of ψ^G to $K^t[H]$ is isomorphic to the direct sum of $\psi(1)$ copies of the regular representation, and therefore ψ^G is faithful on $K^t[H]$. In particular, since

$\psi^G(\mathcal{J}K^t[G] \cap K^t[H]) = 0$, we conclude that $\mathcal{J}K^t[G] \cap K^t[H] = 0$, and the lemma is proved. \square

We close this section with the necessary extension of [18, Lemma 5.4].

Lemma 6.4. *Let $G = SH$ be a finite group with S a normal subgroup of G and with $S/\mathbb{O}_p(S)$ center-by-semisimple. Suppose that, under the permutation action of H on the simple factors of $S/\text{sol } S$, every nonidentity normal subgroup N of H moves at least $4^{|H|}$ points. Then it follows that $\mathcal{J}K^t[G] \cap K^t[H] = 0$.*

Proof. First observe that $S \cap H$ is a normal subgroup of H which normalizes all simple factors of $S/\text{sol } S$. Thus, by assumption, $S \cap H = \langle 1 \rangle$ and hence $G = SH = S \rtimes H$ is a semidirect product. As a consequence, if $P = \mathbb{O}_p(S)$, then P is a normal p -subgroup of G and $G/P = (S/P) \rtimes H$. Furthermore, by Lemma 3.2(ii), $I = \mathcal{J}K^t[P] \cdot K^t[G]$ is an ideal of $K^t[G]$ with $K^t[G]/I = K^t[G/P]$, where the latter is a twisted group algebra of G/P . Note that $H \cap P = \langle 1 \rangle$ implies that $K^t[H]$ embeds faithfully into $K^t[G/P]$ and, of course, $\mathcal{J}K^t[G] \cap K^t[H]$ maps into $\mathcal{J}K^t[G/P] \cap K^t[H]$. Thus it suffices to prove that $\mathcal{J}K^t[G/P] \cap K^t[H] = 0$, or equivalently we can assume that $P = \mathbb{O}_p(S) = \langle 1 \rangle$. In particular, if $Z = \mathbb{Z}(S)$, then Z is a p' -group and, by assumption, $\tilde{S} = S/Z = S/\text{sol } S$ is semisimple.

Following the proof of [18, Lemma 5.4], we now partition the simple factors of \tilde{S} into larger subgroups $\tilde{S}_{i,j}$ via the following somewhat complicated procedure. To start with, let H_1, H_2, \dots, H_r be representatives of the conjugacy classes of subgroups of H and, for each i , let $\{1 = h_{i,1}, h_{i,2}, \dots, h_{i,n_i}\}$ be a full set of right coset representatives for H_i in H . Of course, every transitive permutation representation (H, Γ) of H is isomorphic to a unique $(H, H/H_i)$, where the latter is the permutation action of H on the cosets of H_i .

Now H permutes the set Ω of nonabelian simple factors of \tilde{S} by conjugation and, for each i , let $\Omega_{i,1}, \Omega_{i,2}, \dots, \Omega_{i,v_i}$ be the orbits of this action which are isomorphic to $(H, H/H_i)$. In particular, for each $1 \leq k \leq v_i$, we can choose a, not necessarily unique, simple factor $M_{i,k} \in \Omega_{i,k}$ whose stabilizer in H is equal to H_i . Then $\Omega_{i,k} = \{M_{i,k}^{h_{i,j}} \mid j = 1, 2, \dots, n_i\}$ and, for each i, j , we define $\tilde{S}_{i,j} = \prod_{k=1}^{v_i} M_{i,k}^{h_{i,j}} \triangleleft \tilde{S}$. Note that

1. \tilde{S} is the direct product $\tilde{S} = \prod_{i=1}^r \prod_{j=1}^{n_i} \tilde{S}_{i,j}$,
2. $\tilde{S}_{i,1}$ is stabilized by H_i and $\tilde{S}_{i,j} = \tilde{S}_{i,1}^{h_{i,j}}$,
3. $\tilde{S}_{i,j}$ contains at most one simple factor from each H -orbit of Ω .

Furthermore, if $\mathcal{V} = \{i \mid v_i \geq \log_2 |H|\}$ and if $\tilde{S}_0 = \prod_{i \notin \mathcal{V}} \prod_{j=1}^{n_i} \tilde{S}_{i,j}$, then

4. \tilde{S} is the direct product $\tilde{S} = \tilde{S}_0 \times \prod_{i \in \mathcal{V}} \prod_{j=1}^{n_i} \tilde{S}_{i,j}$.

It is now clear that $G = S \rtimes H$ satisfies most of the hypotheses of Lemma 6.3. To be precise, (i) and (ii) follow from (4) and (2), respectively, while (iv) follows from the fact that each $\tilde{S}_{i,j}$ with $i \in \mathcal{V}$ is a direct product of $v_i \geq \log_2 |H|$ nonabelian

simple groups. Indeed, by Lemma 6.2(i), any twisted group algebra $K^t[\tilde{S}_{i,j}]$ of $\tilde{S}_{i,j}$ is a tensor product of v_i twisted group algebras of nonabelian simple groups, and each of the latter has at least two distinct irreducible representations by Lemma 6.1. Consequently, $K^t[\tilde{S}_{i,j}]$ has at least $2^{v_i} \geq |H|$ irreducible tensor representations, and part (iv) is proved.

Finally, suppose $N \triangleleft H$ is the kernel of the permutation action of H on the set $\{\tilde{S}_{i,j} \mid i \in \mathcal{V}, 1 \leq j \leq n_i\}$. Then N normalizes each $\tilde{S}_{i,j}$ with $i \in \mathcal{V}$ and hence it permutes the simple factors which generate these groups. But, by (3), $\tilde{S}_{i,j}$ contains at most one simple factor from each H -orbit of Ω , and therefore it follows that N must stabilize each such point. In other words, the only simple factors moved by N are those contained in \tilde{S}_0 . But observe that $r \leq 2^{|H|-1}$ and $n_i = |H : H_i| \leq |H|$. Thus, since \tilde{S}_0 is generated by those $\tilde{S}_{i,j}$ which contain $v_i < \log_2 |H|$ simple factors, we conclude that \tilde{S}_0 contains at most

$$(\log_2 |H|) \cdot |H| \cdot 2^{|H|-1} < |H|^2 \cdot 2^{|H|-1} < 2^{|H|+1} \cdot 2^{|H|-1} = 4^{|H|}$$

points of Ω . Consequently, N moves less than $4^{|H|}$ points of Ω and therefore, by assumption, $N = \langle 1 \rangle$. In other words, $G = S \rtimes H$ satisfies all the hypotheses of Lemma 6.3, and we conclude from the latter result that $\mathcal{J}K^t[G] \cap K^t[H] = 0$. \square

§7. THEOREMS 1.3 AND 1.4

As usual, K is a field of characteristic $p > 0$, but now we no longer assume that it is algebraically closed. In this section, we complete our work on central, locally subnormal subgroups. To start with, we offer the

Proof of Theorem 1.4. Recall that G is assumed to be a locally finite group having no nonidentity locally subnormal subgroup, and our aim is to show that $K^t[G]$ is semiprimitive. If \tilde{K} denotes the algebraic closure of K , then $\tilde{K} \otimes K^t[G] = \tilde{K}^t[G]$ is a twisted group algebra of G over \tilde{K} , and $\mathcal{J}K^t[G] \subseteq \mathcal{J}\tilde{K}^t[G]$ by [12, Theorem 7.2.13]. Thus it suffices to show that $\mathcal{J}\tilde{K}^t[G] = 0$ or equivalently, we can now assume that K is algebraically closed. Suppose, by way of contradiction, that $\mathcal{J}K^t[G] \neq 0$. Then, by Lemma 2.1 and [18, Lemma 2.5], there exists a countable subgroup \tilde{G} of G such that \tilde{G} has no nonidentity locally subnormal subgroup and $\mathcal{J}K^t[\tilde{G}] \neq 0$. In other words, \tilde{G} is also a counterexample, so it suffices to assume that $G = \tilde{G}$ is countable and, in particular, that G is the ascending union of the finite groups $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$.

Since $\mathcal{J}K^t[G] \neq 0$, there exists an element $0 \neq \alpha \in \mathcal{J}K^t[G]$ and finite subgroups $\langle \text{supp } \alpha \rangle = A \subseteq B$ which satisfy the conclusion of Lemma 3.3. Furthermore, if $H = A^{[B]}$ is the subnormal closure of A in B , then Proposition 3.4 applies. In particular, part (ii) of the latter result asserts that if T is any subgroup of G normalized by H and if $\mathcal{J}K^t[T] = 0$, then $H \subseteq \Delta(TH)$. By deleting a few terms if necessary, we can clearly assume that $H \subseteq L_1$. Note that $H \neq \langle 1 \rangle$ since otherwise we would

have $K^t[H] = K$ and $0 \neq \alpha \in \mathcal{JK}^t[G] \cap K^t[H] \subseteq \mathcal{JK}^t[H] = 0$. Furthermore, by Proposition 3.4(i), $H = \mathbb{O}^{p'}(H)$ is generated by p -elements.

Now, if $S_i = p\text{-rad } L_i$, then H permutes the set Ω_i of simple factors of $S_i/p\text{-sol } L_i$ by conjugation and, for each nonidentity normal subgroup N of H , we let $f_N(i)$ be the number of points of Ω_i moved by N . In this way, we obtain a finite collection of functions $f_N: \mathcal{N} \rightarrow \mathcal{N} \cup \{0\}$ where \mathcal{N} is the set of positive integers and, as is typical of these proofs, there are two cases to consider according to whether these functions are bounded or not. Note that, if \mathcal{M} is a subsequence of \mathcal{N} , then $G = \bigcup_{i \in \mathcal{M}} L_i$.

Case 1. For some $\langle 1 \rangle \neq N \triangleleft H$, the map f_N is bounded on a subsequence of \mathcal{N} .

Proof. By deleting terms, we can assume that f_N is bounded on \mathcal{N} . Let k be an upper bound for the values of f_N , and let $T = N^{[G]}$ be the local subnormal closure of N in G . We apply the various conclusions of Lemma 3.1. To start with, (iii) shows that T has no nonidentity locally subnormal subgroup, and (iv) implies that H normalizes T and that $T = N^T$. Furthermore, by part (ii), $\{N^{[L_i]} \mid i = 1, 2, \dots\}$ is a local system for T and, by deleting terms if necessary, we can assume that T is the ascending union of the subgroups $T_i = N^{[L_i]} \triangleleft\triangleleft L_i$.

Since $T_i \triangleleft\triangleleft L_i$, we know that

$$T_i/p\text{-sol } T_i = T_i/(T_i \cap p\text{-sol } L_i) \cong T_i(p\text{-sol } L_i)/p\text{-sol } L_i \triangleleft\triangleleft L_i/p\text{-sol } L_i.$$

In particular, any minimal subnormal subgroup of $T_i/p\text{-sol } T_i$ is also a minimal subnormal subgroup of $L_i/p\text{-sol } L_i$, so it follows from Lemma 5.3(v) that the group $p\text{-rad } T_i/p\text{-sol } T_i$ is a direct factor of $p\text{-rad } L_i/p\text{-sol } L_i$. More precisely, if Γ_i denotes the set of simple factors of $p\text{-rad } T_i/p\text{-sol } T_i$, then $\Gamma_i \subseteq \Omega_i$. Indeed, since this inclusion respects the action of N , we see that N moves at most k points of Γ_i .

Finally, since $T = N^T$, Lemma 5.5 now implies that $\mathcal{JK}^t[T] = 0$, and therefore we conclude from Proposition 3.4 that $H \subseteq \Delta(TH)$. Hence $N \subseteq T \cap \Delta(TH) \subseteq \Delta(T)$. But $\Delta(T)$ is generated by all finite normal subgroups of T , so $\Delta(T) = \langle 1 \rangle$ and this is a contradiction since $N \neq \langle 1 \rangle$. \square

Case 2. The functions f_N are unbounded on all subsequences of \mathcal{N} .

Proof. Write $h = |H|$ and let $n = n(h)$ be the function defined immediately preceding Proposition 3.4. By [19, Lemma 1.3], there exists a subsequence $\mathcal{M} \subseteq \mathcal{N}$ such that each f_N is strictly increasing on \mathcal{M} . Consequently, $\min_N \{f_N\}$ is unbounded on \mathcal{M} , so there exists a subscript i with

$$f_N(i) \geq 4^h + h^2 n^{12}$$

for all $\langle 1 \rangle \neq N \triangleleft H$. For convenience, write $L = L_i$ and $S = S_i$. If $Q = \mathbb{O}_{pp'}(L) = \mathbb{O}_{pp'}(p\text{-sol } L)$, then S acts on the p' -group $\hat{Q} = Q/\mathbb{O}_p(L)$ and we let $C = \mathbb{C}_S(\hat{Q}) \triangleleft S$. Our first goal is to show that each N moves a large number of simple factors of $C(p\text{-sol } L)/p\text{-sol } L$. For this, we need the following observation.

Claim. *If x is a p -element of H , then x moves at most hn^{12} simple factors of the semisimple group $S/C(p\text{-sol } L)$.*

Proof. Since $H \subseteq L_1 \subseteq L$, we see that H normalizes Q . Thus, since $\hat{Q} = Q/\mathbb{O}_p(L)$ is a p' -group, Proposition 3.4(iii) implies that $H\mathbb{O}_p(L)/\mathbb{O}_p(L) \subseteq \Delta_n(QH/\mathbb{O}_p(L))$ and, in particular, that $|\hat{Q} : \mathbb{C}_{\hat{Q}}(x)| \leq n = n(h)$. Now H normalizes S and the group SH acts on \hat{Q} . Thus, there is a natural homomorphism $\tilde{\cdot} : SH \rightarrow SH/\mathbb{C}_{SH}(\hat{Q})$, and the image acts faithfully on \hat{Q} . Since \tilde{x} is a p -element of \widetilde{SH} , it follows that \tilde{x} normalizes a Sylow p -subgroup \tilde{P} of \tilde{S} . Consequently, if we set $\tilde{M} = \mathbb{N}_{\tilde{S}}(\tilde{P}) \subseteq \mathbb{N}_{\widetilde{SH}}(\tilde{P})$, then the inequality $|\hat{Q} : \mathbb{C}_{\hat{Q}}(\tilde{x})| \leq n$ and Lemma 4.3(ii) imply that $[[\tilde{P}, \tilde{x}]^{\tilde{M}}, \tilde{x}]$ has order at most $(n^4!)^{n^4}$.

Now $\tilde{S} = S/C(p\text{-sol } L)$ is a homomorphic image of $\tilde{S} = S/C$ and therefore \tilde{P} maps onto \bar{P} , a Sylow p -subgroup of \bar{S} . Moreover, \tilde{M} maps onto $\bar{M} = \mathbb{N}_{\bar{S}}(\bar{P})$. Thus, since x normalizes \bar{P} , we conclude from the above that $[[\bar{P}, x]^{\bar{M}}, x]$ has order at most $(n^4!)^{n^4}$. Furthermore, since \bar{S} is also a homomorphic image of $S/p\text{-sol } L$, it follows that \bar{S} is semisimple and hence that $\bar{S} = \bar{U}_1 \times \bar{U}_2 \times \cdots \times \bar{U}_r$ is a direct product of nonabelian simple groups. Thus $\bar{P} = \bar{P}_1 \times \bar{P}_2 \times \cdots \times \bar{P}_r$ and $\bar{M} = \bar{M}_1 \times \bar{M}_2 \times \cdots \times \bar{M}_r$, where \bar{P}_i is a Sylow p -subgroup of \bar{U}_i and where $\bar{M}_i = \mathbb{N}_{\bar{U}_i}(\bar{P}_i)$.

Of course, $\langle x \rangle$ permutes the simple factors \bar{U}_i and x normalizes \bar{P} , so $\langle x \rangle$ permutes the factors $\bar{P}_i = \bar{P} \cap \bar{U}_i$ of \bar{P} . As we will see below, each nontrivial orbit under the action of $\langle x \rangle$ yields a nontrivial contribution to the commutator group $[[\bar{P}, x]^{\bar{M}}, x]$. The argument is somewhat easier when $p \geq 3$ since the \bar{M} -term is not needed, but we will treat all primes in a uniform manner using the normalizer \bar{M} .

To start with, since S is the p -radical of L , it follows that all simple factors of $S/p\text{-sol } L$ have order divisible by p . Thus, the same is true of the simple factors of \bar{S} , and therefore each \bar{P}_i is a nonidentity p -group. Suppose, for example, that \bar{U}_1 is contained in a nontrivial $\langle x \rangle$ -orbit and say that $\bar{U}_1^x = \bar{U}_2$. Since \bar{P}_1 cannot be contained in the center of its normalizer \bar{M}_1 in \bar{U}_1 , by Burnside's Theorem, we can choose $\bar{y} \in \bar{P}_1$ and $\bar{z} \in \bar{M}_1$ with $[\bar{y}^{-1}, \bar{z}] \neq 1$. Then $[\bar{y}, x] = \bar{y}^{-1}\bar{y}^x \in (\bar{P}_1 \times \bar{P}_2) \cap [\bar{P}, x]$, so $[[\bar{y}, x], \bar{z}] = [\bar{y}^{-1}, \bar{z}]$ is a nonidentity element in $\bar{P}_1 \cap [\bar{P}, x]^{\bar{M}}$. Hence $[[\bar{y}^{-1}, \bar{z}], x]$ is a nonidentity element in $(\bar{P}_1 \times \bar{P}_2) \cap [[\bar{P}, x]^{\bar{M}}, x]$, and this observation is proved. It now follows that if $\langle x \rangle$ has t nontrivial orbits, then $[[\bar{P}, x]^{\bar{M}}, x]$ has order at least p^t . In other words, $p^t \leq (n^4!)^{n^4}$ so, since $m! \leq m^m$ for any integer m , we have

$$t \leq n^4 \log_p n^4! \leq n^8 \log_p n^4 \leq n^{12}.$$

But each orbit of $\langle x \rangle$ has size at most $|\langle x \rangle| \leq |H| = h$, and therefore x moves at most hn^{12} points. Thus, the claim is proved. \square

Now H contains at most $|H| = h$ elements of p -power order, so it follows from the preceding claim that these elements move at most h^2n^{12} simple factors of the

semisimple group $S/C(p\text{-sol } L)$. But H is generated by p -elements, so any simple factor moved by H must be moved by some p -element $x \in H$. In other words, we conclude that H moves at most $h^2 n^{12}$ simple factors of $S/C(p\text{-sol } L)$ and, of course, the same holds true for any subgroup N of H .

Next observe that, since $S/p\text{-sol } L$ is semisimple, we have

$$\frac{S}{p\text{-sol } L} \cong \frac{S}{C(p\text{-sol } L)} \times \frac{C(p\text{-sol } L)}{p\text{-sol } L} \cong \frac{S}{C(p\text{-sol } L)} \times \frac{C}{C \cap p\text{-sol } L}$$

and that these isomorphisms respect the action of H . In particular, since each $\langle 1 \rangle \neq N \triangleleft H$ moves at most $h^2 n^{12}$ simple factors of $S/C(p\text{-sol } L)$ and at least $f_N(i) \geq 4^h + h^2 n^{12}$ simple factors of $S/p\text{-sol } L$, it follows that each such N moves at least 4^h simple factors of $C/(C \cap p\text{-sol } L)$.

Finally, note that $C \cap p\text{-sol } L$ is a subgroup of the p -solvable group $p\text{-sol } L$ which centralizes $\hat{Q} = \mathbb{O}_{pp'}(p\text{-sol } L)/\mathbb{O}_p(p\text{-sol } L)$. Thus, [5, Lemma 1.2.3] implies that $C \cap p\text{-sol } L \subseteq \mathbb{O}_{pp'}(p\text{-sol } L) = Q$. In particular, since C centralizes \hat{Q} and since $C/(C \cap p\text{-sol } L)$ is semisimple, it follows that $C/\mathbb{O}_p(C)$ is center-by-semisimple. Furthermore, as we observed previously, H normalizes C , and all nonidentity normal subgroups N of H move at least $4^{|H|}$ simple factors of $C/(C \cap p\text{-sol } L) = C/\text{sol } C$. Thus, Lemma 6.4 applies to the group CH and yields $\mathcal{J}K^t[CH] \cap K^t[H] = 0$. But $\alpha \in \mathcal{J}K^t[G] \cap K^t[CH] \subseteq \mathcal{J}K^t[CH]$, so $0 \neq \alpha \in \mathcal{J}K^t[CH] \cap K^t[H] = 0$, and this contradiction proves the result. \square

As we mentioned in the Introduction, the proof of this critical theorem uses $p\text{-rad } L/p\text{-sol } L$ rather than the group $\text{rad } L/\text{sol } L$ of [18] to guarantee that each simple factor of the semisimple quotient has order divisible by p . This may seem like a fairly weak conclusion, but it was clearly a crucial ingredient in the preceding argument. The remainder of the results in this section now follow quite quickly. For convenience, we let $\mathbb{S}(G)$ denote the characteristic subgroup of G generated by all locally subnormal subgroups of G .

Lemma 7.1. *If Z is a central subgroup of G , then $\mathbb{S}(G/Z) = \mathbb{S}(G)/Z$.*

Proof. Since locally subnormal subgroups map to locally subnormal subgroups, we have $\mathbb{S}(G)/Z \subseteq \mathbb{S}(G/Z)$. Conversely, let A/Z be a locally subnormal subgroup of G/Z . Then A is a center-by-finite subgroup of G , so A is an f.c. group and hence A is generated by its finite normal subgroups. But if B is such a finite normal subgroup of A , then $B \triangleleft A$ and $A/Z \text{ lsn } G/Z$ clearly imply that $B \text{ lsn } G$. Hence $A \subseteq \mathbb{S}(G)$ and therefore $\mathbb{S}(G)$ maps onto $\mathbb{S}(G/Z)$. \square

Of course, if $H \triangleleft G$ then $\mathbb{S}(H) = \mathbb{S}(G) \cap H$. We can now prove

Corollary 7.2. *Let $K^t[G]$ be a twisted group algebra of the locally finite group G over the field K of characteristic $p > 0$. If all locally subnormal subgroups of G are central p' -groups, then $K^t[G]$ is semiprimitive.*

Proof. By assumption, $S = \mathbb{S}(G)$ is a central p' -group. In particular, if $x \in S$, then $\mathbb{C}_G(x) = G$, so $\mathbb{C}_G^t(x) \triangleleft G$ with $G/\mathbb{C}_G^t(x)$ a cyclic p' -group. Consequently, $H = \mathbb{C}_G^t(S) = \bigcap_{x \in S} \mathbb{C}_G^t(x)$ is a normal subgroup of G with G/H a p' -group. We first show that $K^t[H]$ is semiprimitive.

To this end, observe that $\mathbb{S}(H) = \mathbb{S}(G) \cap H = S \cap H$ and that $\mathbb{C}_H^t(S \cap H) = H$. In other words, $K^t[S \cap H]$ is a central subalgebra of $K^t[H]$, and it is semiprimitive since $S \cap H$ is a p' -group. In particular, if \mathcal{M} denotes the set of maximal ideals of $K^t[S \cap H]$, then we have $\bigcap_{I \in \mathcal{M}} I = 0$. Furthermore, since $K^t[S \cap H]$ is central in $K^t[H]$, it follows that $I \cdot K^t[H]$ is an ideal of $K^t[H]$, and freeness implies that $\bigcap_{I \in \mathcal{M}} I \cdot K^t[H] = 0$. Now if $I \in \mathcal{M}$, then $K^t[S \cap H]/I = F_I$ is some field extension of K . Thus, by considering coset representatives for $S \cap H$ in H , it follows that $K^t[H]/I \cdot K^t[H] = F_I^t[H/(S \cap H)]$ is a suitable twisted group algebra of the group $H/(S \cap H)$ over F_I . But $\mathbb{S}(H/(S \cap H)) = \langle 1 \rangle$ by the previous lemma, so Theorem 1.4 implies that $F_I^t[H/(S \cap H)] = K^t[H]/I \cdot K^t[H]$ is semiprimitive. As a consequence, since $\bigcap_{I \in \mathcal{M}} I \cdot K^t[H] = 0$, we conclude that $\mathcal{J}K^t[H] = 0$.

Finally, let $\alpha \in \mathcal{J}K^t[G]$ and define $A = H\langle \text{supp } \alpha \rangle$. Then A/H is a finite p' -group, so Proposition 2.2 implies that $\mathcal{J}K^t[A] = \mathcal{J}K^t[H] \cdot K^t[A] = 0$. But $\alpha \in \mathcal{J}K^t[G] \cap K^t[A] \subseteq \mathcal{J}K^t[A]$, so $\alpha = 0$ and hence $\mathcal{J}K^t[G] = 0$ as required. Alternately, we could apply [16, Lemma 3.4(iii)] to obtain the same conclusion. \square

Note that, even if we wished to obtain the above corollary just for ordinary group algebras, twisted group algebras would nevertheless come into play in the proof. The following is the twisted version of Theorem 1.3. It requires K to be algebraically closed since otherwise Lemma 3.2 would not apply.

Theorem 7.3. *Let G be a locally finite group in which all locally subnormal subgroups are central, and let K be an algebraically closed field of characteristic $p > 0$. If $Z = \mathbb{Z}(G)$ and P is the p -primary part of the center Z , then*

$$\mathcal{J}K^t[G] = \mathcal{J}K^t[Z] \cdot K^t[G] = \mathcal{J}K^t[P] \cdot K^t[G]$$

where $\mathcal{J}K^t[P]$ is a characteristic ideal of $K^t[P]$ of codimension 1. In particular, $K^t[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p .

Proof. Since K is an algebraically closed field and P is a normal p -subgroup of G , Lemma 3.2 implies that $\mathcal{J}K^t[P]$ is a characteristic ideal of $K^t[P]$ of codimension 1 and that $I = \mathcal{J}K^t[P] \cdot K^t[G]$ is an ideal of $K^t[G]$ contained in $\mathcal{J}K^t[G]$. Furthermore, $K^t[G]/I \cong K^t[G/P]$ where the latter is a suitable twisted group algebra of G/P . Now, by assumption, $\mathbb{S}(G) = Z$ and therefore, by Lemma 7.1, $\mathbb{S}(G/P) = Z/P$ is a central p' -subgroup of G/P . Thus Corollary 7.2 implies that $\mathcal{J}K^t[G/P] = 0$ and we conclude that $I = \mathcal{J}K^t[G]$, as required. Finally, $\mathcal{J}K^t[Z] = \mathcal{J}K^t[P] \cdot K^t[Z]$ since Z/P is a p' -group, so the result follows. \square

In the case of ordinary group algebras, Lemma 3.2 clearly holds without the assumption that K is algebraically closed. Indeed, here $\mathcal{J}K[P]$ is the augmentation ideal of $K[P]$, and therefore the above argument also proves Theorem 1.3.

§8. FINITARY LINEAR GROUPS

To proceed further, we will need two results of interest on locally finite, finitary groups. To start with, we say that G acts in a *finitary manner* on the group V if $|V : \mathbb{C}_V(x)| < \infty$ for all $x \in G$. Furthermore, G acts in a **-finitary manner* if the action is finitary and if all G -stable subgroups of V are normal in V . In particular, both of these concepts include the usual notion of a finitary action of a group G on a vector space V over a finite field. Notice that we do not assume, at this point, that G acts faithfully on V . Note further that if G is *-finitary on V and if W is a G -stable subgroup of V , then G acts in a *-finitary manner on both W and V/W . Of course, G acts *irreducibly* on V if V has no proper G -stable subgroup.

Lemma 8.1. *Let G act in a *-finitary manner on the group V , and assume that $G = H^G$ is the normal closure of some finite subgroup H . If T is a fixed normal subgroup of G with $T = T'$, then V has a finite chain*

$$\langle 1 \rangle = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = V$$

of G -stable subgroups such that, for each i , either G acts irreducibly on V_{i+1}/V_i or T acts trivially on V_{i+1}/V_i .

Proof. Since G acts in a *-finitary manner on V and since H is a finite subgroup of G , it follows that $|V : \mathbb{C}_V(H)| = n < \infty$, and we proceed by induction on n . Of course, if $n = 1$ then H centralizes V and hence so does $T \subseteq G = H^G$. Now suppose that $n > 1$ and that the result holds for all such situations with parameter less than n .

We first observe that $X_1 = \mathbb{C}_V(T)$ is a G -stable subgroup of V and that T acts trivially on X_1 . Furthermore, if X_2/X_1 is the centralizer of T on V/X_1 , then T stabilizes the chain $\langle 1 \rangle \subseteq X_1 \subseteq X_2$ and acts trivially on each factor. Thus $T' = T$ acts trivially on X_2 , and it follows from [7, Theorem 8.32] that $X_1 = X_2$. In other words, we can now mod out by X_1 and assume that $\mathbb{C}_V(T) = \langle 1 \rangle$.

Similarly, observe that $Y_1 = [V, T]$ is a G -stable subgroup of V and that T acts trivially on V/Y_1 . Furthermore, if $Y_2 = [Y_1, T]$, then T stabilizes the chain $\langle 1 \rangle \subseteq Y_1/Y_2 \subseteq V/Y_2$ and acts trivially on each factor. Again, this implies that $T' = T$ acts trivially on V/Y_2 , and therefore $Y_2 = Y_1$. As above, we can now replace V by $Y_1 = [V, T]$ and assume that T acts nontrivially on every nonidentity G -stable factor of V .

Now if G acts irreducibly on V , then we are done. Thus, we can suppose that W is a nontrivial G -stable subgroup of V , and we write $C = \mathbb{C}_V(H)$. Since $\mathbb{C}_W(H) = C \cap W$, we have

$$|W : \mathbb{C}_W(H)| = |W : C \cap W| = |WC : C| \leq |V : C| = n.$$

Furthermore, if equality occurs, then $V = WC$ and H acts trivially on V/W . But then $T \subseteq G = H^G$ also acts trivially on the factor V/W and this contradicts our previous assumption. In other words, $|W : \mathbb{C}_W(H)| < n$, and induction implies that W contains an appropriate chain of G -stable subgroups.

Similarly, observe that $CW/W \subseteq \mathbb{C}_{V/W}(H)$, and therefore

$$|V/W : \mathbb{C}_{V/W}(H)| \leq |V/W : CW/W| = |V : CW| \leq |V : C| = n.$$

Furthermore, if equality occurs, then $CW = C$, so $W \subseteq C$ and H acts trivially on W . Again this implies that $T \subseteq G = H^G$ acts trivially on W , a contradiction. Thus $|V/W : \mathbb{C}_{V/W}(H)| < n$ and, this time, induction implies that V/W has an appropriate chain of G -stable subgroups. \square

For convenience, we say that a locally finite group V is *semisimple* if it is a direct product of finite nonabelian simple groups. Again, any normal subgroup of V is a partial direct product of these factors and, consequently, the minimal normal subgroups of V are precisely the nonabelian simple factors.

Lemma 8.2. *Let $G = H^G$, where H is a finite subgroup of G , and suppose that G acts faithfully and in a $*$ -finitary manner on a locally finite f.c. group V . If G acts irreducibly on V , then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$$

with each quotient G_{i+1}/G_i either infinite simple, locally solvable, or an f.c. group. Furthermore, each such infinite simple group is a finitary linear group over a finite field $GF(q)$ for some prime q involved in V .

Proof. We can assume that $V \neq \langle 1 \rangle$. Since V is an f.c. group, it has a finite, minimal normal subgroup N which is either an elementary abelian q -group for some prime q , or a direct product of nonabelian simple groups. Since any two distinct minimal normal subgroups of V commute, and since $V = N^G$, it follows that V is either an elementary abelian q -group, or it is semisimple.

Suppose first that V is an elementary abelian q -group, so that G is an irreducible finitary linear group over the field $GF(q)$. If G is primitive, then it follows from [20] (or see [21, Theorem 10.12.2]) that G is either finite or simple-by-locally solvable. Thus, the result follows in this case.

Next, if G is imprimitive, then we can write $V = \prod_{i \in \mathcal{I}} V_i$ as a nontrivial direct product of subgroups permuted transitively by G . Since $G = H^G$, H must act nontrivially in its permutation action on the factors V_i and hence it has a nontrivial orbit. But, if this orbit contains V_1 , then certainly $V_1 \cap \mathbb{C}_V(H) = \langle 1 \rangle$, and hence all V_i are finite. Furthermore, since $[V, H]$ is finite it follows easily that H centralizes all but finitely many of the V_i 's. Consequently, G is a faithful permutation group on the set $\Omega = \bigcup_{i \in \mathcal{I}} V_i$, and H acts in a finitary manner on this set. Thus $G = H^G \subseteq$

FSym_Ω , and [18, Proposition 4.1] yields the result, since the finitary alternating groups FAlt_Γ are finitary linear groups over any field.

Finally, if V is semisimple, then $V = \prod_{i \in \mathcal{I}} V_i$ is a direct product of finite non-abelian simple groups. Moreover, since the V_i 's are the minimal normal subgroups of V , they are permuted transitively by G . Again, G acts faithfully as a finitary permutation group on the set $\Omega = \bigcup_{i \in \mathcal{I}} V_i$, so the result follows as above. \square

With this, we can now obtain our first key observation on finitary groups.

Proposition 8.3. *Let G act faithfully and in a $*$ -finitary manner on the locally finite f.c. group V . If G is the normal closure of a finite subgroup H , then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$$

with each quotient G_{i+1}/G_i either infinite simple, locally solvable, or an f.c. group. Furthermore, each such infinite simple group is a finitary linear group over a finite field $GF(q)$ for some prime q involved in V .

Proof. Let T be the intersection of all normal subgroups X of G with G/X locally solvable. Then G/T embeds in the full direct product $\prod_X G/X$ and, since G is locally finite, it follows easily that G/T is locally solvable. Thus T is the unique smallest normal subgroup of G with quotient which is locally solvable and, consequently, $T = T'$. By Lemma 8.1, V has a finite chain

$$\langle 1 \rangle = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = V$$

of G -stable subgroups such that, for each i , either G acts irreducibly on V_{i+1}/V_i or T acts trivially on V_{i+1}/V_i . We prove the proposition by induction on k , the case $k = 0$ being obvious.

Suppose now that $k \geq 1$ and let $N \triangleleft G$ be the kernel of the action of G on V/V_1 . Then, by induction, G/N has a finite subnormal series with factors as described above. Next, let $\tilde{\cdot}: G \rightarrow G/\mathcal{C}_G(V_1)$ be the natural epimorphism. Then \tilde{G} acts faithfully on V_1 and certainly $\tilde{G} = \tilde{H}^{\tilde{G}}$. If \tilde{G} acts irreducibly on V_1 , then the preceding lemma implies that \tilde{G} has an appropriate subnormal series. On the other hand, if T acts trivially on V_1 , then \tilde{G} is locally solvable and again the group has a suitable series. Furthermore, since $\tilde{N} \triangleleft \tilde{G}$, it follows that \tilde{N} also has a finite subnormal series with factors which are either infinite simple, locally solvable, or f.c. groups. Finally, observe that $\mathcal{C}_N(V_1)$ stabilizes the chain $\langle 1 \rangle \subseteq V_1 \subseteq V$ and acts trivially on each factor. Thus, since $\mathcal{C}_N(V_1)$ acts faithfully on V , it must be abelian and, by combining this fact with the subnormal series for G/N and for $\tilde{N} = N/\mathcal{C}_N(V_1)$, the result follows. \square

The second key fact on finitary groups is [13, Lemma 2.10], namely

Lemma 8.4. *Let G be a locally finite, locally p -solvable group acting faithfully and in a finitary manner on a p' -group V . If \tilde{G} is any nonidentity homomorphic image of G , then either $\mathbb{O}_p(\tilde{G}) \neq \langle 1 \rangle$ or $\mathbb{O}_{p'}(\tilde{G}) \neq \langle 1 \rangle$.*

Proof. If x is a p -element of G , then $|V : \mathbb{C}_V(x)| = n < \infty$. Thus, since G acts faithfully on V , it follows from Lemma 4.3(i) that the commutator group $[[P, x], x]$ has order at most $(n^2!)^{n^2}$, for all finite p -subgroups P of G containing x . Now, obviously, this latter condition is inherited by homomorphic images. In other words, if \tilde{G} is a homomorphic image of G and if \tilde{x} is a p -element of \tilde{G} , then the commutator group $[[\tilde{P}, \tilde{x}], \tilde{x}]$ has order bounded by a function of \tilde{x} , for all finite p -subgroups \tilde{P} of \tilde{G} containing \tilde{x} .

Finally, suppose $\mathbb{O}_p(\tilde{G}) = \langle 1 \rangle = \mathbb{O}_{p'}(\tilde{G})$. Since \tilde{G} is locally p -solvable, Lemma 4.6 clearly implies that $\tilde{H} = \mathbb{T}_p(\tilde{H})$ for every finite subgroup \tilde{H} of \tilde{G} . But, if \tilde{L} is any finite subgroup of \tilde{G} , then it follows from Lemma 4.4 that there exists a finite $\tilde{H} \supseteq \tilde{L}$ with $\tilde{L} \cap \mathbb{T}_p(\tilde{H}) = \langle 1 \rangle$. In other words, $\tilde{L} = \tilde{L} \cap \tilde{H} = \langle 1 \rangle$, and therefore $\tilde{G} = \langle 1 \rangle$, as required. \square

We close this section with the complete semiprimitivity theorem for locally p -solvable groups.

Proposition 8.5. *Let $K^t[G]$ be a twisted group algebra of the locally finite, locally p -solvable group G over the field K of characteristic $p > 0$. If G has no locally subnormal subgroup of order divisible by p , then $\mathcal{J}K^t[G] = 0$.*

Proof. As usual, let $S = \mathbb{S}(G)$ be the subgroup of G generated by all locally subnormal subgroups. By assumption, S is a p' -group, and hence $K^t[S]$ is semiprimitive. In particular, if $C = \mathbb{C}_G(S)$ and $D = \mathbb{D}_G(S)$, then Proposition 3.5 implies that $\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G]$. In other words, it suffices to show that $\mathcal{J}K^t[D] = 0$. Note that D/C is a locally finite, locally p -solvable group having a faithful finitary action on the p' -group S . Thus, the preceding lemma applies to D/C .

Observe that $\mathbb{S}(C) = C \cap S = \mathbb{Z}(S)$. Thus, all locally subnormal subgroups of C are central p' -groups, and therefore Corollary 7.2 implies that $K^t[C]$ is semiprimitive. Now consider the collection \mathcal{U} of all normal subgroups U of D , containing C , with $\mathcal{J}K^t[U] = 0$. Then $C \in \mathcal{U}$, so \mathcal{U} is nonempty. Furthermore, if $\{U_i \mid i \in \mathcal{I}\}$ is a nonempty chain in \mathcal{U} , then $U = \bigcup_{i \in \mathcal{I}} U_i$ is a subgroup of D with $C \subseteq U \triangleleft D$ and with $\mathcal{J}K^t[U] = 0$. To see the latter, let $\alpha \in \mathcal{J}K^t[U]$. Since $\text{supp } \alpha$ is finite, it therefore follows that $\alpha \in K^t[U_i]$ for some i , and hence $\alpha \in \mathcal{J}K^t[U] \cap K^t[U_i] \subseteq \mathcal{J}K^t[U_i] = 0$. By Zorn's Lemma, we can now choose U to be a maximal member of \mathcal{U} .

If $U \neq D$, then D/U is a nontrivial homomorphic image of D/C and hence, by Lemma 8.4, either $\mathbb{O}_p(D/U) \neq \langle 1 \rangle$ or $\mathbb{O}_{p'}(D/U) \neq \langle 1 \rangle$. In other words, there exists $U \subset V \triangleleft D$ with V/U either a p -group or a p' -group. Of course, $\mathbb{S}(V) = V \cap \mathbb{S}(D) = V \cap S$, so V has no locally subnormal subgroup of order divisible by p . Furthermore, note that any locally finite p -group is locally nilpotent and hence generated by its

locally subnormal subgroups. Thus, since $\mathcal{J}K^t[U] = 0$, [16, Lemma 3.4(iii)] implies that $\mathcal{J}K^t[V] = 0$. But, this contradicts the maximality of U , so $D = U \in \mathcal{U}$ and the proposition is proved. \square

§9. THEOREM 1.5

In this final section, we prove our main results on the semiprimitivity problem. To start with, we need the twisted analog of [18, Proposition 3.1]. Note that the transitivity of subnormality implies that if $A \text{ lsn } N$ and $N \triangleleft G$, then $A \text{ lsn } G$.

Proposition 9.1. *Let $K^t[G]$ be the twisted group algebra of a locally finite group G over a field K of characteristic $p > 0$. Suppose that G has a finite subnormal series*

$$G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient G_{i+1}/G_i either

- i. locally p -solvable, or
- ii. an infinite simple group, or
- iii. generated by its locally subnormal subgroups.

If $\mathcal{J}K^t[G_0] = 0$ and G has no locally subnormal subgroup of order divisible by p , then $K^t[G]$ is semiprimitive.

Proof. We proceed by induction on n , the case $n = 0$ being given. Now suppose that the result holds for $n - 1$ and set $H = G_{n-1} \triangleleft G$. Since H has no locally subnormal subgroup of order divisible by p , the inductive assumption implies that $\mathcal{J}K^t[H] = 0$. Furthermore [16, Lemma 3.4(iii)] implies that $\mathcal{J}K^t[G] = 0$ if $G/H = G_n/G_{n-1}$ is generated by its locally subnormal subgroups. Thus we can assume that G/H is either locally p -solvable or infinite simple.

Now $\mathcal{J}K^t[H] = 0$, so Proposition 3.5 implies that $\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G]$ where $D = \mathbb{D}_G(H)$, and it suffices to show that $\mathcal{J}K^t[D] = 0$. To this end, note that

$$N = D \cap H = \mathbb{D}_G(H) \cap H = \mathbb{D}_H(H) = \Delta(H)$$

is an f.c. group. Thus, since $N \triangleleft G$ has no locally subnormal subgroup of order divisible by p , it follows that N is a p' -group and hence that $\mathcal{J}K^t[N] = 0$ by Lemma 2.1. Furthermore,

$$D/N = D/(D \cap H) \cong DH/H \triangleleft G/H.$$

In particular, if G/H is locally p -solvable, then D is also locally p -solvable and Proposition 8.5 yields the result. On the other hand, if G/H is infinite simple, then either $D = N$ or D/N is infinite simple. In the former case, $\mathcal{J}K^t[D] = \mathcal{J}K^t[N] = 0$ as required, and thus it suffices to assume that D/N is infinite simple. For this, note that D acts in a finitary manner on H , so it certainly acts in a finitary manner

on $N = H \cap D$. In other words, $D = \mathbb{D}_D(N)$, and therefore the result follows from [16, Proposition 3.3]. \square

It is interesting to compare the above to the more elementary argument in Proposition 4.9. We can now quickly prove the twisted version of Theorem 1.5.

Theorem 9.2. *Let $K^t[G]$ be a twisted group algebra of the locally finite group G over the field K of characteristic $p > 0$. If G has no locally subnormal subgroup of order divisible by p , then $K^t[G]$ is semiprimitive.*

Proof. Again, let $S = \mathbb{S}(G)$ be the subgroup of G generated by all locally subnormal subgroups. By assumption, S is a p' -group, and hence $K^t[S]$ is semiprimitive. In particular, Proposition 3.5 implies that $\mathcal{J}K^t[G] = \mathcal{J}K^t[D] \cdot K^t[G]$, where $D = \mathbb{D}_G(S)$. In other words, it suffices to show that $\mathcal{J}K^t[D] = 0$.

Suppose, by way of contradiction, that $\mathcal{J}K^t[D] \neq 0$ and choose $0 \neq \alpha \in \mathcal{J}K^t[D]$. Set $H = \langle \text{supp } \alpha \rangle$ and let $T = H^{[D]}$ be the local subnormal closure of H in D . Of course, $\alpha \in \mathcal{J}K^t[D] \cap K^t[T] \subseteq \mathcal{J}K^t[T]$, so $K^t[T]$ is not semiprimitive. Furthermore, Lemma 3.1 implies that T has no locally subnormal subgroup of order divisible by p and that $T = H^T$ is the normal closure of H in T .

Now D acts in a finitary manner on S , so certainly T acts in a finitary manner on $V = \mathbb{S}(T) \subseteq S$. Furthermore, since $T \supseteq V$, we conclude that V is an f.c. group and that T acts in a $*$ -finitary manner on V . Thus, since $T = H^T$, it is easy to see that Proposition 8.3 applies to T/C , where $C = \mathbb{C}_T(V)$. In particular, T has a finite subnormal series

$$C = T_0 \triangleleft T_1 \triangleleft \cdots \triangleleft T_m = T$$

where each factor T_{i+1}/T_i is either infinite simple, locally solvable, or an f.c. group. Of course, any f.c. group is generated by its locally subnormal subgroups.

Finally, $\mathbb{S}(C) = V \cap C = \mathbb{Z}(V)$ is a p' -group, so Corollary 7.2 implies that $\mathcal{J}K^t[T_0] = \mathcal{J}K^t[C] = 0$. Thus, since T has no locally subnormal subgroup of order divisible by p , Proposition 9.1 implies that $\mathcal{J}K^t[T] = 0$, and this contradiction yields the result. \square

Theorem 1.5 now follows immediately from Lemma 1.1 and the preceding result. Furthermore, we note that if K is a perfect field of characteristic $p > 0$ and if A is a finite group, then $\mathcal{J}K^t[A] = 0$ if and only if p does not divide the order of A . Thus, a complete twisted analog of Theorem 1.5 holds in the case of perfect fields. Finally, Theorem 9.2 and [12, Theorems 9.2.4 and 9.2.5] yield

Corollary 9.3. *Let $K^t[G]$ be a twisted group algebra of the countable, locally finite group G over the field K of characteristic $p > 0$. If G has no locally subnormal subgroup of order divisible by p and if $\Delta(G) = \langle 1 \rangle$, then $K^t[G]$ is a primitive ring.*

In other words, in the above situation, $K^t[G]$ has a faithful irreducible representation. As we mentioned previously, it still remains to precisely describe the

Jacobson radical $\mathcal{JK}^t[G]$ when G is an arbitrary locally finite group. Here, we conjecture that

$$\mathcal{JK}^t[G] = \mathcal{JK}^t[\mathbb{S}(G)] \cdot K^t[G]$$

when $\mathbb{O}_p(G) = \langle 1 \rangle$. Since $\mathbb{S}(G)$ is generated by the locally subnormal subgroups of G , it is quite easy to describe $\mathcal{JK}^t[\mathbb{S}(G)]$, and therefore the above *controlling* formula would certainly settle the problem. Note that Theorem 1.3 confirms the conjecture when $\mathbb{S}(G)$ is assumed to be central.

REFERENCES

1. E. Artin, *The orders of the linear groups*, Comm. Pure Applied Math. **8** (1955), 355–366.
2. D. Gorenstein, *Finite Simple Groups*, Plenum, New York, 1982.
3. J. I. Hall, *Locally finite simple groups of finitary linear transformations*, Proceedings of the Istanbul NATO ASI Conference on Finite and Locally Finite Groups, Kluwer, Dordrecht, 1995 (to appear).
4. ———, *Periodic simple groups of finitary linear transformations* (to appear).
5. P. Hall and G. Higman, *The p -length of a p -soluble group and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3) **6** (1956), 1–42.
6. R. B. Howlett and I. M. Isaacs, *On groups of central type*, Math. Zeit **179** (1982), 555–569.
7. I. M. Isaacs, *Algebra: A Graduate Course*, Brooks/Cole, Pacific Grove, 1994.
8. G. Karpilovsky, *Projective Representations of Finite Groups*, Dekker, New York, 1985.
9. O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups*, North-Holland, Amsterdam, 1973.
10. D. S. Passman, *Radicals of twisted group rings*, Proc. London Math. Soc. (3) **20** (1970), 409–437.
11. ———, *A new radical for group rings?*, J. Algebra **28** (1974), 556–572.
12. ———, *The Algebraic Structure of Group Rings*, Wiley-Interscience, New York, 1977.
13. ———, *The Jacobson radical of a group ring of a locally solvable group*, Proc. London Math. Soc. **38** (1979), 169–192, (ibid **39** (1979), 208–210).
14. ———, *Semiprimitivity of group algebras of locally finite groups*, Infinite Groups and Group Rings, World Scientific, Singapore, 1993, pp. 77–101.
15. ———, *Semiprimitivity of group algebras of infinite simple groups of Lie Type*, Proc. Amer. Math. Soc. **121** (1994), 399–403.
16. ———, *Semiprimitivity of group algebras of locally finite groups II*, J. Pure Appl. Algebra (to appear).
17. ———, *Semiprimitivity of group algebras: a survey*, Atas da XIII Escola de Algebra, IMECC-UNICAMP, pp. 168–187.

18. ———, *The semiprimitivity problem for group algebras of locally finite groups*, Israel J. Math. (to appear).
19. D. S. Passman and A. E. Zalesskiĭ, *Semiprimitivity of group algebras of locally finite simple groups*, Proc. London Math. Soc. **67** (1993), 243–276.
20. R. E. Phillips, *Primitive, locally finite, finitary linear groups* (to appear).
21. ———, *Finitary linear groups: a survey*, Proceedings of the Istanbul NATO ASI Conference on Finite and Locally Finite Groups, Kluwer, Dordrecht, 1995 (to appear).
22. H. Wielandt, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Zeit. **45** (1939), 209–244.
23. ———, *Unendliche Permutationsgruppen*, Lecture Notes, Mathematisches Institut der Universität, Tübingen, 1960.
24. ———, *Mathematical Works*, Volume 1, de Gruyter, Berlin, 1994.
25. H. Zassenhaus, *The Theory of Groups*, Second Edition, Chelsea, New York, 1958.

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