SIMPLE LIE ALGEBRAS OF WITT-TYPE

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Abstract. Let K be a field, let A be an associative, commutative K-algebra and let Δ be a nonzero K-vector space of commuting K-derivations of A. Then, with a rather natural definition, $A \otimes_K \Delta = A\Delta$ becomes a Lie algebra, and we obtain necessary and sufficient conditions here for this Lie algebra to be simple. With one minor exception in characteristic 2, simplicity occurs if and only if A is Δ-simple and $A^\Delta \otimes \Delta = A^\Delta \Delta$ acts faithfully as derivations on A.

§1. Introduction

Let K be a field, let A be an associative, commutative K-algebra and let Δ be a nonzero K-vector space of commuting K-derivations of A. Then the tensor product $A \otimes_K \Delta = A\Delta$ acts on A by way of

$$a \otimes \partial : x \mapsto a\partial(x) \qquad \text{for all } a, x \in A,\ \partial \in \Delta.$$

Since A is commutative, this gives rise to a linear transformation

$$\theta : A\Delta \to \operatorname{Der}_K(A) \subseteq \operatorname{Hom}_K(A, A).$$

Furthermore, suppose $a_1, a_2 \in A$ and $\partial_1, \partial_2 \in \Delta$. Then, since ∂_1 and ∂_2 commute, we obtain the equality

$$a_1\partial_1 \cdot a_2\partial_2 - a_2\partial_2 \cdot a_1\partial_1 = a_1\partial_1(a_2)\partial_2 - a_2\partial_2(a_1)\partial_1$$

as operators on A. Consequently, the image of θ is a Lie subalgebra of $\operatorname{Der}_K(A)$. Indeed, the preceding displayed equation motivates the definition of the binary operation $[\ ,\ ]$ on $A\Delta$ as the K-linear extension of

$$[a_1\partial_1, a_2\partial_2] = a_1\partial_1(a_2)\partial_2 - a_2\partial_2(a_1)\partial_1 \qquad \text{for all } a_1, a_2 \in A,\ \partial_1, \partial_2 \in \Delta.$$

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As is well known, this yields a Lie algebra structure on $A\Delta$ and then θ is clearly a Lie homomorphism.

The standard example of this construction is the Witt algebra $W_n = A\Delta$. Here $A = K[t_1^{\pm}, t_2^{\pm}, \ldots, t_n^{\pm}]$ is the ring of Laurent polynomials in the variables $t_1, t_2, \ldots, t_n$, and Δ is the K-vector space spanned by the partial derivatives $\partial/\partial t_i$. More general versions of this construction have been considered by Kaplansky [Kp], Kawamoto [Kw], Osborn [O], Đoković and Zhao [DZ], and others. The goal of this paper is to look at an arbitrary $A \otimes \Delta = A\Delta$ from a ring-theoretic point of view and to obtain necessary and sufficient conditions for this Lie algebra to be simple.

One obvious necessary condition is that A be Δ-simple. This means that A has no nontrivial Δ-stable ideals. Indeed, if I is a Δ-stable ideal of A, then $I \otimes \Delta = I\Delta$ is clearly a Lie ideal of $A\Delta$. Hence, if $A\Delta$ is simple, then I must 0 or A. Note that the Δ-simplicity of A also forces the ring of constant A^Δ to have a rather nice structure.

Lemma 1.1. *If A is Δ-simple, then A^Δ is a field containing K.*

Proof. We know that A^Δ is a subring of A. Furthermore, if $0 \neq a \in A^\Delta$, then aA is a nonzero Δ-stable ideal of A and hence $aA = A$. Thus a is invertible in A and, since $\partial(1/a) = -\partial(a)/a^2 = 0$, it follows that $1/a \in A^\Delta$. □

A second necessary condition for the simplicity of $A\Delta$ is that this Lie algebra act faithfully on A. Indeed, if $A\Delta$ is simple, then the kernel of the Lie homomorphism θ is either 0 or $A\Delta$. But, in the latter case, $1\Delta = \Delta$ acts trivially on A, contradicting its definition as a nonzero subspace of $\mathrm{Der}_K(A)$. Thus $A\Delta$ must act faithfully and hence so must its subspace $A^\Delta \otimes \Delta = A^\Delta\Delta$. As we see below, all that is required is this weaker assumption. Specifically, we will prove

Theorem 1.2. *Let A be a commutative K-algebra and let Δ be a nonzero K-vector space of commuting K-derivations of A. Assume that either* $\mathrm{char}\, K \neq 2$ *or* $\mathrm{char}\, K = 2$ *and* $\dim_K \Delta \geq 2$. *Then $A\Delta = A \otimes \Delta$ is a simple Lie algebra if and only if A is Δ-simple and $A^\Delta\Delta$ acts faithfully on A.*

As is apparent, there is one missing case here. When $\dim_K \Delta = 1$, then it is easy to see from Lemma 1.1 that $A^\Delta\Delta$ acts faithfully on A. Thus this condition can be dropped, but when $\mathrm{char}\, K = 2$, another condition comes into play. Indeed, we will also show

Theorem 1.3. *Let A be a commutative K-algebra and let Δ be a K-vector space of K-derivations of A. Assume that* $\mathrm{char}\, K = 2$ *and that* $\dim_K \Delta = 1$. *Then $A\Delta = A \otimes \Delta$ is a simple Lie algebra if and only if A is Δ-simple and $\Delta(A) = A$.*

Note that, if A is a field, then A is automatically Δ-simple. Furthermore, if $A^\Delta = K$, then $A^\Delta\Delta = \Delta$ and, by assumption, Δ acts faithfully on A. Thus we obtain

Corollary 1.4. *Let A be a field extension of K and let Δ be a nonzero K-vector space of commuting derivations of A. If $A^\Delta = K$, then $A\Delta$ is a simple Lie algebra unless* $\operatorname{char} K = 2$, $\dim_K \Delta = 1$ *and* $\Delta(A) \neq A$.

In the exceptional case, the answer can go either way, as can be seen by the following examples. Observe that if $\operatorname{char} K = p > 0$, then all pth powers of elements of A are constants. Consequently, A is algebraic over A^Δ.

Lemma 1.5. *Let K be a field of characteristic $p > 0$ and let $x_0, x_1, x_2, \dots$ be elements which are algebraically independent over K.*

i. *If A is the finitely generated field $A = K(x_0, x_1, \dots, x_n)$ and if ∂ is the derivation of A defined by $\partial(x_0) = 1$ and $\partial(x_i) = x_{i-1}$ for $1 \leq i \leq n$, then ∂ is not onto. In particular, if $p = 2$ and $\Delta = K\partial$, then $A\Delta$ is not simple.*
ii. *If A is the infinitely generated field $A = K(x_0, x_1, x_2, \dots)$ and if ∂ is the derivation of A defined by $\partial(x_0) = 1$ and $\partial(x_i) = x_{i-1}$ for $i \geq 1$, then ∂ is onto. Thus, if $p = 2$ and $\Delta = K\partial$, then $A\Delta$ is a simple Lie algebra.*

Proof. (i) Since A is algebraic over $A^\Delta \supseteq K$, we have $(A : A^\Delta) < \infty$. Furthermore, ∂ is an A^Δ-linear transformation which is not one-to-one. Hence ∂ is not onto and Theorem 1.3 yields the result when $\operatorname{char} K = 2$.

(ii) Again, A is algebraic over $A^\Delta \supseteq K$ and hence $A = A^\Delta[x_0, x_1, x_2, \dots]$. With this, it is clear that ∂ is a locally nilpotent operator on A. Furthermore, since $1 \in A$ is infinitely integrable, repeated integration by parts shows that ∂ is onto. Indeed, if $\nu(a) = \sum_{m=0}^{\infty} (-)^m x_m \partial^m(a)$, then $\partial\nu(a) = a$ for all $a \in A$. As above, Theorem 1.3 yields the result on the characteristic 2 example. □

We thank Jim Osterburg for suggesting the use of integration by parts here. Alternating, we could have used the methods of [KN] to construct epimorphic derivations of fields. Finally, we indicate how these results relate to the work of [Kw] and others. To start with, let G be a multiplicative abelian group and let $A = K[G]$ be the group algebra of G over K. If $\lambda \in \operatorname{Hom}(G, K^+)$, then λ gives rise to a K-linear operator $\lambda^\#: A \to A$ defined by $\lambda^\#: \sum_{g \in G} k_g g \mapsto \sum_{g \in G} k_g \lambda(g) g$. It is easy to verify that $\lambda^\# \in \operatorname{Der}_K(A)$ and that $\lambda \neq 0$ implies that $\lambda^\# \neq 0$. In particular, if Λ is a nonzero K-subspace of $\operatorname{Hom}(G, K^+)$, then $\Lambda^\# = \Delta$ is a nonzero K-subspace of $\operatorname{Der}_K(A)$ which is easily seen to consist of commuting derivations. Thus, it makes sense to consider the Lie algebra $A\Delta$.

For convenience, we write $G^\Lambda = \{ g \in G \mid \lambda(g) = 0 \text{ for all } \lambda \in \Lambda \}$. Then G^Λ is certainly a subgroup of G and G/G^Λ is isomorphic to a subgroup of a (complete) direct product of copies of K^+. Thus, G/G^Λ is torsion-free abelian if $\operatorname{char} K = 0$ and it is an elementary abelian p-group if $\operatorname{char} K = p > 0$. The following result generalizes [Kw; Corollary 3.2].

Corollary 1.6. *Let G be a multiplicative abelian group, let $A = K[G]$ and let Λ be a nonzero K-subspace of* $\operatorname{Hom}(G, K^+)$. *Then $\Delta = \Lambda^\#$ is a nonzero K-vector space* *of commuting derivations of A and $A \otimes \Delta = A\Delta$ is a simple Lie algebra if and only if $G^\Lambda = \langle 1 \rangle$ and $\dim_K \Lambda \geq 2$ when $\operatorname{char} K = 2$.*

Proof. It is clear that $A^\Delta = K[G]^\Delta = K[G^\Lambda]$.

If $A\Delta$ is simple, then A is Δ-simple and hence, by Lemma 1.1, $A^\Delta = K[G^\Lambda]$ is a field. But group algebras have augmentation ideals, so $K[G^\Lambda]$ is a field if and only if $G^\Lambda = \langle 1 \rangle$. Furthermore, if $\lambda \in \Lambda$, then $\lambda^\#\colon A \to A$ cannot be onto since $\lambda(1) = 0$. Hence, if $\operatorname{char} K = 2$, then we must have $\dim_K \Lambda = \dim_K \Delta \geq 2$.

Conversely, suppose $G^\Lambda = \langle 1 \rangle$ and that $\dim_K \Lambda = \dim_K \Delta \geq 2$ when $\operatorname{char} K = 2$. Then $A^\Delta = K[G^\Lambda] = K$, so certainly $A^\Delta \Delta = \Delta$ acts faithfully on A. Furthermore, suppose $I \neq 0$ is a Δ-stable ideal of A and let α be a nonzero element of I of minimal support size. By replacing α by $g\alpha$ if necessary, we can assume that $1 \in \operatorname{supp} \alpha$. But then, $\lambda^\#(\alpha) \in I$ has smaller support since $\lambda(1) = 0$, so $\lambda^\#(\alpha) = 0$ for all $\lambda \in \Lambda$. Hence $\alpha \in A^\Delta = K$, so α is a unit and $I = A$. We conclude that A is Δ-simple and the result now follows from Theorem 1.2. □

A more general framework for such examples is as follows. Again, let G be a multiplicative abelian group and let $A = \oplus \sum_{g \in G} A_g$ be a commutative G-graded K-algebra. Thus, each A_g is a K-subspace of A, $A_x A_y \subseteq A_{xy}$ for all $x, y \in G$, and $1 \in A_1$. For convenience, we say that A is *annihilator-free* if $\operatorname{ann}_{A_g} A_{g^{-1}} = 0$ for all $g \in G$. In particular, this holds if $A = A_1[G]$ is an ordinary group algebra since each $A_{g^{-1}}$ contains a unit. It, of course, also holds when A is a domain and all components A_g are nonzero.

Corollary 1.7. *Let G be a multiplicative abelian group and let $A = \oplus \sum_{g \in G} A_g$ be an annihilator-free, G-graded commutative K-algebra. Suppose that Δ is a nonzero K-vector space of commuting derivations of A and that Δ stabilizes each A_g.*

- i. *A is Δ-simple if and only if A_1 is Δ-simple and A^Δ is a field.*
- ii. *Assume that either $\operatorname{char} K \neq 2$ or $\operatorname{char} K = 2$ and $\dim_K \Delta \geq 2$. If $A_1 \otimes \Delta = A_1\Delta$ is Lie simple and $A^\Delta = A_1^\Delta$, then $A \otimes \Delta = A\Delta$ is Lie simple.*

Proof. (i) Suppose first that A is Δ-simple. Then, by Lemma 1.1, we know that A^Δ is a field. Also if J is a Δ-stable ideal of A_1, then it is clear that JA is a Δ-stable ideal of A and hence $JA = 0$ or A. Since the G-graded structure $A = \oplus \sum_{g \in G} A_g$ implies that $JA \cap A_1 = J$, we conclude that $J = 0$ or A_1, as required.

Conversely, suppose that A_1 is Δ-simple and that A^Δ is a field. Let I be a nonzero Δ-stable ideal of A and let $a = \sum_{g \in G} a_g \in \sum_{g \in G} A_g$ be a nonzero element of I of minimal support size. If $a_x \neq 0$, then $a_x \notin \operatorname{ann}_{A_x} A_{x^{-1}} = 0$, so there exists an element $b_{x^{-1}} \in A_{x^{-1}}$ with $0 \neq a_x b_{x^{-1}} \in A_1$. Replacing a by $ab_{x^{-1}} \in I$ if necessary, we can now assume that $a_1 \neq 0$.

Let S denote the support of a and let

$$J = \Big\{ c_1 \in A_1 \ \Big|\ c = \sum_{g \in S} c_g \in I \Big\}.$$

Then it is clear that J is a nonzero ideal of A_1 and that J is Δ-stable since each A_g is Δ-stable. But A_1 is Δ-simple, so $J = A_1$ and we can now choose $c \in I$, as above, with $c_1 = 1$. Finally, if $\partial \in \Delta$, then $\partial(c) \in I$. Moreover, the support of $\partial(c)$ is properly smaller than S since $\partial(1) = 0$. Thus the minimal nature of S implies that $\partial(c) = 0$. Indeed, since this is true for all such ∂, we have $0 \neq c \in I \cap A^{\Delta}$. But A^{Δ} is a field, so c is a unit in A and consequently $I = A$.

(ii) Since $A_1 \otimes \Delta = A_1\Delta$ is Lie simple, Theorem 1.2 implies that $A_1^{\Delta}\Delta$ acts faithfully on A_1 and that A_1 is Δ-simple. In particular, A_1^{Δ} is a field by Lemma 1.1. But $A^{\Delta} = A_1^{\Delta}$, by assumption, so A^{Δ} is a field. It now follows from (i) above that A is Δ-simple. Finally, $A^{\Delta}\Delta = A_1^{\Delta}\Delta$ acts faithfully on A_1, so $A^{\Delta}\Delta$ acts faithfully on A. With this, Theorem 1.2 yields the result. □

Examples of this nature appear in [IKN1, IKN2] and are special cases of the following more general construction. Let B be any commutative, associative K-algebra and let Δ be a K-vector space of commuting derivations of B. If $\mathcal{G}$ is any additive subgroup of B, then by taking formal exponentials, we obtain the multiplicative group $G = e^{\mathcal{G}} = \{\, e^g \mid g \in \mathcal{G} \,\}$. Now let $A = B[G]$ be the group algebra of G over B, and extend the action of Δ to A by defining

$$\partial(be^g) = \big(\partial(b) + b\partial(g)\big)e^g \qquad \text{for all } b \in B,\ g \in \mathcal{G},\ \partial \in \Delta.$$

Then, it is easy to see that Δ is now a K-vector space of commuting derivations of A. Furthermore, we say that B and $G = e^{\mathcal{G}}$ are disjoint if B contains no element which behaves like e^g with $0 \neq g \in \mathcal{G}$. Since $a = e^g$ satisfies $\partial(a) = \partial(g)a$ for all $\partial \in \Delta$, the precise disjointness condition here is that if $b \in B$ and $g \in \mathcal{G}$ satisfy $\partial(b) = \partial(g)b$ for all $\partial \in \Delta$, then either $b = 0$ or $g = 0$.

Corollary 1.8. *Let B, Δ and $\mathcal{G}$ be as above, with B and $G = e^{\mathcal{G}}$ disjoint, and set $A = B[G] = B[e^{\mathcal{G}}]$. If $B\Delta$ is Lie simple and if either* $\operatorname{char} K \neq 2$ *or* $\operatorname{char} K = 2$ *and* $\dim_K \Delta \geq 2$*, then $A\Delta$ is also Lie simple.*

Proof. In view of Corollary 1.7(ii), it suffices to show that $A^{\Delta} = B^{\Delta}$. To this end, let $a = \sum_{g \in \mathcal{G}} b_g e^g \in A^{\Delta}$. Since Δ stabilizes each component Be^g, it follows that $0 = \partial(b_g e^g) = \big(\partial(b_g) + b_g\partial(g)\big)e^g$, and hence $\partial(b_g) = \partial(-g)b_g$ for all $\partial \in \Delta$. But $-g \in \mathcal{G}$, so disjointness implies that either $b_g = 0$ or $g = 0$. In other words, $a = b_0e^0 = b_0 \in B^{\Delta}$, and the result follows. □

Finally, suppose that K is the field of real or complex numbers, that B is a ring of infinitely differentiable functions in the variables $x_1, x_2, \ldots$, and that Δ is the K-linear span of the partial derivatives $\partial/\partial x_i$ for $i = 1, 2, \ldots$. Then, by viewing the formal exponentials as ordinary exponential functions, we obtain an algebra homomorphism from $A = B[G]$ into the ring of infinitely differentiable functions in $x_1, x_2, \ldots$. But the kernel of this map is surely a Δ-stable ideal of A, so since A is Δ-simple in the above situation, it follows that A is naturally isomorphic to the ring of functions generated by B and $\exp(\mathcal{G})$.

§2. Proofs

In this section, we prove the sufficiency part of the main theorems. Thus, we assume throughout that A is a commutative, associative K-algebra and that Δ is a nonzero K-vector space of commuting K-derivations of A. Furthermore, we suppose that

i. A is Δ-simple, and that
ii. $A^{\Delta}\Delta$ acts faithfully on A.

Finally, we let L be a nonzero Lie ideal of $A \otimes \Delta = A\Delta$. In view of Lemma 1.1, we know that A^{Δ} is a field extension of K.

If V is a K-subspace of A, then the sum of all ideals of A contained in V is the unique largest ideal of A contained in V. In particular, the ideal I mentioned in part (i) of the following lemma always exists.

Lemma 2.1. *Let V be a Δ-stable, K-subspace of A.*

i. *If I is the largest ideal of A contained in V, then I is Δ-stable. In particular, if V contains a nonzero ideal, then $V = A$.*
ii. *VA is a Δ-stable ideal of A. Thus $V \neq 0$ implies that $VA = A$.*

Proof. (i) Let $\partial \in \Delta$. Then $I + \partial(I) \subseteq V$ and $I + \partial(I) \triangleleft A$ since, for all $a \in A$ and $x \in I$, we have $a\partial(x) = -\partial(a)x + \partial(ax) \in I + \partial(I)$. Thus $I + \partial(I) \subseteq I$ by the maximal nature of I, and hence I is Δ-stable. Finally, if V contains a nonzero ideal, then I is a nonzero Δ-stable ideal of A. But A is Δ-simple, so $I = A$ and therefore $V = A$.

(ii) Certainly $VA \triangleleft A$ and VA is Δ-stable since both V and A are Δ-stable. In particular, if $V \neq 0$, then the Δ-simplicity of A implies that $VA = A$. □

Such subspaces V are of interest because of

Lemma 2.2. *Let $\partial_1, \partial_2, \ldots, \partial_n$ be K-linearly independent elements of Δ and let V be the set of all elements $a \in A$ such that there exists $\alpha = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n \in L$ with $a = a_n$. Then V is a Δ-stable, K-subspace of A.*

Proof. Since L is a K-subspace of $A\Delta$, it is clear that V is a K-subspace of A. Furthermore, if $\partial \in D$, then $1\partial \in A\Delta$, and $\partial(a_1)\partial_1 + \partial(a_2)\partial_2 + \cdots + \partial(a_n)\partial_n = [1\partial, \alpha] \in L$ since $L \triangleleft A\Delta$. Thus $\partial(a) = \partial(a_n) \in V$ and V is Δ-stable. □

It is now a simple matter to prove

Proposition 2.3. *$A\Delta$ acts faithfully on A.*

Proof. Let $L \triangleleft A\Delta$ be the kernel of the action of $A\Delta$ on A and assume, by way of contradiction, that $L \neq 0$. Then we can let $n \geq 1$ be the minimal support size of a nonzero element of L. In other words, there exist linearly independent elements $\partial_1, \partial_2, \ldots, \partial_n$ in Δ with $L \cap \sum_{i=1}^{n} A\partial_i \neq 0$ and such that all nonzero elements in this intersection have all their A-coefficients nonzero. Let V be defined as in the previous lemma. Then V is a nonzero Δ-stable, K-subspace of A. Furthermore, the nature of the action of $A\Delta$ on A implies that $AL \subseteq L$ and hence $AV \subseteq V$. Thus $V \triangleleft A$ and, since A is Δ-simple, we have $V = A$. In particular, $1 \in V$ and we can find $\alpha = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n \in L$ with $a_n = 1$. If $\partial \in \Delta$, then $\partial(a_1)\partial_1 + \partial(a_2)\partial_2 + \cdots + \partial(a_{n-1})\partial_{n-1} = [1\partial, \alpha] \in L$ since $\partial(a_n) = \partial(1) = 0$. Thus the minimality of n implies that $\partial(a_i) = 0$ for all i and all $\partial \in \Delta$. In other words, $a_i \in A^\Delta$ for all i, and $\alpha \in A^\Delta\Delta$. But, by assumption, $A^\Delta\Delta$ acts faithfully on A, so we have the required contradiction. We conclude, therefore, that $L = 0$ and hence that $A\Delta$ acts faithfully on A. $\square$

Recall that L is a fixed nonzero Lie ideal of $A\Delta$. As a consequence of the above, we obtain

Lemma 2.4. *If ∂ is a nonzero element of Δ, then $L \cap A\partial \neq 0$.*

Proof. Since $L \neq 0$, we can choose $n \geq 1$ minimal so that there exist linearly independent elements $\partial_1, \partial_2, \ldots, \partial_n \in \Delta$ with $\partial = \partial_1$ and $L \cap \sum_{i=1}^n A\partial_i \neq 0$. The goal is to show that $n = 1$. Thus suppose that $n \geq 2$ and let V be defined as in Lemma 2.2. Then V is a Δ-stable, K-subspace of A and the minimality of n implies that $V \neq 0$.

Let $\alpha = \sum_{i=1}^n a_i\partial_i$ be a nonzero element of $L \cap \sum_{i=1}^n A\partial_i$. If $x \in A$, then $[\alpha, x\partial_1] \in L$ and it is clear that this element has support in $\{\, \partial_1, \partial_2, \ldots, \partial_n \,\}$. Thus we can write $[\alpha, x\partial_1] = \sum_{i=1}^n c_i(x)\partial_i$. The most complicated coefficient is that of ∂_1. Specifically, $c_1(x) = -x\partial_1(a_1) + \sum_{i=1}^n a_i\partial_i(x)$ and consequently $c_1(A) \neq 0$. Indeed, if $c_1(A) = 0$, then $x = 1$ implies that $\partial_1(a_1) = 0$ and hence $0 = c_1(x) = \sum_{i=1}^n a_i\partial_i(x) = \alpha(x)$. In other words, α acts trivially on A, so $\alpha = 0$ by the previous proposition, a contradiction.

We have therefore shown that $[\alpha, x\partial_1]$ is not identically 0, so the minimal nature of $n \geq 2$ implies that $c_n(A) \neq 0$. Here $c_n(x) = -x\partial_1(a_n)$, so $\partial_1(a_n) \neq 0$ and V contains the nonzero ideal $A\partial_1(a_n)$. It now follows from Lemma 2.1(i) that $V = A$. But then we could have chosen α so that $a_n = 1$ and hence $\partial_1(a_n) = 0$, a contradiction. Thus $n = 1$ and $0 \neq L \cap A\partial_1 = L \cap A\partial$, as required. $\square$

With this result in hand, one expects the main theorems to follow quite easily and this is indeed the case. We begin with $\operatorname{char} K \neq 2$.

Lemma 2.5. *Let $\operatorname{char} K \neq 2$. If ∂ is any nonzero element of Δ, then $L \supseteq A\partial$. Thus $L = A\Delta$ and $A\Delta$ is a simple Lie algebra.*

Proof. Let $V = \{\, a \in A \mid a\partial \in L \,\}$. Then, by Lemma 2.2, V is a Δ-stable, K-subspace of A. Furthermore, the preceding lemma implies that $V \neq 0$. If $v \in V$ and $y \in A$, then $\big[v\partial(y) - y\partial(v)\big]\partial = [v\partial, y\partial] \in L$, so

$$2v\partial(y) - \partial(vy) = v\partial(y) - y\partial(v) \in V \qquad \text{for all } v \in V,\ y \in A. \tag{1}$$

In particular, replacing y by vy, we have

$$v^2\partial(y) \in V \qquad \text{for all } v \in V,\ y \in A. \tag{2}$$

Fix $a \in V$ and $b \in A$. Then, by equation (2), we have $a^2\partial(b) \in V$. Also $\partial(b^2/2) = \partial(b)b$, so $a^2\partial(b)b \in V$. Using $v = a^2\partial(b)$ and $y = bx$ in (1), we get

$$2a^2\partial(b)\partial(bx) - \partial(a^2\partial(b)bx) \in V \qquad \text{for all } x \in A. \tag{3}$$

Similarly, if we use $v = a^2\partial(b)b$ and $y = x$ in (1), we get

$$2a^2\partial(b)b\partial(x) - \partial(a^2\partial(b)bx) \in V \qquad \text{for all } x \in A. \tag{4}$$

Next, we subtract expression (4) from expression (3) to obtain

$$2a^2\partial(b)\big[\partial(bx) - b\partial(x)\big] = 2a^2\partial(b)^2x \in V \qquad \text{for all } x \in A.$$

In other words, $V \supseteq a^2\partial(b)^2A$ for all $a \in V,\ b \in A$.

It remains to show that we can choose suitable a and b with $a^2\partial(b)^2 \neq 0$. To this end, note that $V \neq 0$ so $VA = A$ by Lemma 2.1(ii). Similarly, $\partial(A) \neq 0$ and this is a Δ-stable, K-subspace of A since the derivations in Δ commute. Thus $\partial(A)A = A$ by Lemma 2.1(ii) again, and hence $A = VA\cdot\partial(A)A = V\partial(A)A$. It follows that not all generators of this ideal can be nilpotent. Thus there exist $a \in V$ and $\partial(b) \in \partial(A)$ with $a\partial(b)$ not nilpotent. In particular, $a^2\partial(b)^2 \neq 0$, so V contains the nonzero ideal $a^2\partial(b)^2A$. By Lemma 2.1(i), we conclude that $V = A$, as required. □

This, of course, yields Theorem 1.2 when $\operatorname{char} K \neq 2$, so it remains to consider fields of characteristic 2. In this situation, all the minus signs in the formulas become plus.

Lemma 2.6. *Let ∂_1 and ∂_2 be linearly independent elements of Δ.*

- i. *If $\partial_1(a) \neq 0$, then there exists $b \in A$ with $\partial_1(a)\partial_2(b) + \partial_2(a)\partial_1(b) \neq 0$.*
- ii. *The composition $\partial_1\partial_2$ is not zero.*

Proof. (i) Since $\partial_1(a) \neq 0$, we see that $\alpha = \partial_1(a)\partial_2 + \partial_2(a)\partial_1$ is a nonzero element of $A\Delta$. Thus, since $A\Delta$ acts faithfully on A by Proposition 2.3, there exists $b \in A$ with $\partial_1(a)\partial_2(b) + \partial_2(a)\partial_1(b) = \alpha(b) \neq 0$.

(ii) Since $\partial_1 \neq 0$, we can choose $a \in A$ with $\partial_1(a) \neq 0$, and let $b \in A$ be given by part (i) above. Then

$$\partial_1\partial_2(ab) = \big[\partial_1(a)\partial_2(b) + \partial_2(a)\partial_1(b)\big] + \partial_1\partial_2(a)\cdot b + a\cdot\partial_1\partial_2(b).$$

But $\partial_1(a)\partial_2(b) + \partial_2(a)\partial_1(b) \neq 0$, so it follows that $\partial_1\partial_2(x) \neq 0$ for $x = a$ or b or ab. Thus $\partial_1\partial_2 \neq 0$. □

Recall that L is a fixed nonzero Lie ideal of $A\Delta$.

Lemma 2.7. *Let* char $K = 2$ *and let* ∂ *be a nonzero element of* Δ*. Then* $L \cap A\partial \supseteq \partial(A)\partial = [A\partial, A\partial]$*. In particular, if* $\dim_K \Delta = 1$*, then* $A\Delta$ *is simple if and only if* $\Delta(A) = A$*.*

Proof. Say $L \cap A\partial = V\partial$. Then it follows from Lemmas 2.2 and 2.4 that V is a nonzero Δ-stable, K-subspace of A. Hence $VA = A$ by Lemma 2.1(ii). Now if $a, b \in A$, then

$$[a\partial, b\partial] = (a\partial(b) + b\partial(a))\partial = \partial(ab)\partial.$$

Thus $[A\partial, A\partial] = \partial(A)\partial$. Also, if $a \in V$ and $b \in A$, then $[a\partial, b\partial] \in L$, so the above implies that $\partial(ab) \in V$. In other words, $V \supseteq \partial(VA) = \partial(A)$, so the first part is proved.

Finally, suppose $\Delta = K\partial$ is 1-dimensional so that $\Delta(A) = \partial(A)$. If $\partial(A) = A$, then the above implies that $L \supseteq A\partial = A\Delta$ and $A\Delta$ is simple. Conversely, if $\partial(A)$ is properly smaller than A, then $[A\partial, A\partial]$ is a nonzero ideal of $A\partial$ properly smaller than $A\partial$ and hence $A\partial = A\Delta$ is not simple. $\square$

If $\partial \neq 0$, then it is clear that $A^{\Delta}\partial$ acts faithfully on A. Thus the preceding result proves Theorem 1.3. It therefore remains to consider the case where char $K = 2$ and $\dim_K \Delta \geq 2$. To start with, we need

Lemma 2.8. *If* char $K = 2$*, then* $L \supseteq [A\Delta, A\Delta]$*.*

Proof. Fix $0 \neq \partial \in \Delta$. Let $\partial_1 \in D \setminus K\partial$ and let V be the set of ∂_1-coefficients in $L \cap (A\partial + A\partial_1)$. By Lemma 2.6(ii), there exists an element $a \in A$ with $\partial\partial_1(a) \neq 0$ and set $b = \partial_1(a)$. Then $b\partial_1 \in L \cap A\partial_1$ by the previous lemma, so for all $x \in A$ we have $b\partial_1(x)\partial + x\partial(b)\partial_1 = [x\partial, b\partial_1] \in L$. In other words, V contains the ideal $A\partial(b)$ and note that $\partial(b) = \partial\partial_1(a) \neq 0$. We conclude from Lemmas 2.2 and 2.1(i) that $V = A$ and this clearly implies that $A\partial + L \supseteq A\partial_1$. Since the latter inclusion holds for all such ∂_1, it follows that $A\partial + L = A\Delta$. Hence $A\Delta/L \cong A\partial/(A\partial \cap L)$ and the latter Lie algebra is abelian by Lemma 2.7. $\square$

Finally, we can prove

Lemma 2.9. *Let* char $K = 2$ *and* $\dim_K \Delta \geq 2$*. If* ∂ *is a nonzero element of* Δ*, then* $L \cap A\partial = A\partial$*. In particular,* $L = A\Delta$ *and* $A\Delta$ *is a simple Lie algebra.*

Proof. Since $\dim D \geq 2$, we can choose $\partial_1 \in \Delta \setminus K\partial$. Let $a, x \in A$. Since $L \supseteq [A\Delta, A\Delta]$ by the previous lemma, it follows that L contains

$$\begin{aligned}[a\partial, x\partial_1] + [1\partial, ax\partial_1] &= x\partial_1(a)\partial + a\partial(x)\partial_1 + \partial(ax)\partial_1 \\ &= x\partial_1(a)\partial + x\partial(a)\partial_1.\end{aligned}$$

In particular, replacing x by $x\partial(b)$ for some $b \in A$, we see that L contains the element $x\partial_1(a)\partial(b)\partial + x\partial(a)\partial(b)\partial_1$. Furthermore, by interchanging a and b and by adding the two expressions, we conclude that L contains $x\big(\partial_1(a)\partial(b) + \partial(a)\partial_1(b)\big)\partial$.

Now ∂ and ∂_1 are linearly independent, so Lemma 2.6(i) implies that we can choose a and b with $c = \partial_1(a)\partial(b) + \partial(a)\partial_1(b) \neq 0$. Thus $L \cap A\partial \supseteq (Ac)\partial$ and, since Ac is a nonzero ideal of A, it follows from Lemmas 2.2 and 2.1(i) that $L \cap A\partial = A\partial$. In particular, since this is true for all such ∂, we conclude that $L = A\Delta$ and hence that $A\Delta$ is a simple Lie algebra. □

Obviously, Lemmas 2.5 and 2.9 combine to yield Theorem 1.2.

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