Poisson representations of measure-valued processes

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with Eliane Rodrigues
Poisson random measures

$S$ a Polish space and $\nu$ be a $\sigma$-finite measure on $\mathcal{B}(S)$.

$\xi$ is a Poisson random measure with mean measure $\nu$ if

a) $\xi$ is a random counting measure on $S$.

b) For each $A \in \mathcal{S}$ with $\nu(A) < \infty$, $\xi(A)$ is Poisson distributed with parameter $\nu(A)$.

c) For $A_1, A_2, \ldots \in \mathcal{S}$ disjoint, $\xi(A_1), \xi(A_2), \ldots$ are independent.

Lemma 1 If $H : S \to S_0$, Borel measurable, and $\tilde{\xi}(A) = \xi(H^{-1}(A))$, then $\tilde{\xi}$ is a Poisson random measure on $S_0$ with mean measure $\tilde{\nu}$ given by $\tilde{\nu}(A) = \nu(H^{-1}(A))$.
Moment identities

If $\xi$ is a Poisson random measure with mean measure $\nu$

$$E[e^{\int f(z)\xi(dz)}] = e^{\int (e^f - 1)d\nu},$$
or letting $\xi = \sum \delta Z_i$,

$$E[\prod g(Z_i)] = e^{\int (g - 1)d\nu}.$$  

Similarly,

$$E[\sum h(Z_j) \prod g(Z_i)] = \int hgd\nu e^{\int (g - 1)d\nu},$$

and

$$E[\sum_{i \neq j} h(Z_i)h(Z_j) \prod g(Z_k)] = (\int hgd\nu)^2 e^{\int (g - 1)d\nu},$$
Conditionally Poisson systems

Let $\xi$ be a random counting measure on $S$ and $\Xi$ be a locally finite random measure on $S$.

$\xi$ is *conditionally Poisson* with Cox measure $\Xi$ if, conditioned on $\Xi$, $\xi$ is a Poisson point process with mean measure $\Xi$.

$$E[e^{-\int_S f d\xi}] = E[e^{-\int_S (1-e^{-f}) d\Xi}]$$

for all nonnegative $f \in M(S)$.

If $\xi$ is conditionally Poisson system on $S \times [0, \infty)$ with Cox measure $\Xi \times m$ where $m$ is Lebesgue measure, then for $f \in M(S)$

$$E[e^{-\int_{S \times [0,K]} f d\xi}] = E[e^{-K \int_S (1-e^{-f}) d\Xi}]$$

and for $f \geq 0$,

$$\Xi(f) = \lim_{K \to \infty} \frac{1}{K} \int_{S \times [0,K]} f d\xi = \lim_{\epsilon \to 0} \epsilon \int_{S \times [0,\infty)} e^{-\epsilon u} f(x) \xi(dx \times du) \ a.s.$$
Relationship to exchangeability

Lemma 2 Suppose $\xi$ is a conditionally Poisson random measure on $S \times [0, \infty)$ with Cox measure $\Xi \times m$. If $\Xi < \infty$ a.s., then we can write $\xi = \sum_{i=1}^{\infty} \delta_{(X_i, U_i)}$ with $U_1 < U_2 < \cdots$ a.s. and $\{X_i\}$ is exchangeable.

Conditioned on $\Xi$, $\{U_i\}$ is Poisson with parameter $\Xi(S)$ and $\{X_i\}$ is iid with distribution $\Xi(S)^{-1} \Xi$. 
Convergence

Lemma 3 If \( \{\xi_n\} \) is a sequence of conditionally Poisson random measures on \( S \times [0, \infty) \) with Cox measures \( \{\Xi_n \times m\} \). Then \( \xi_n \Rightarrow \xi \) if and only if \( \Xi_n \Rightarrow \Xi \), \( \Xi \) the Cox measure for \( \xi \).

If \( \xi_n \rightarrow \xi \) in probability, then \( \Xi_n \rightarrow \Xi \) in probability
A population model

Consider a process with state space \( E = \cup_n [0, r]^n \).

\( 0 \leq g \leq 1 \) and \( f(u, n) = \prod_{i=1}^{n} g(u_i) \)

For \( a > 0 \), and \( -\infty < b \leq ra \), define

\[
Af(u, n) = f(u, n) \sum_{i=1}^{n} 2a \int_{u_i}^{r} (g(v) - 1) dv + f(u, n) \sum_{i=1}^{n} (au_i^2 - bu_i) \frac{g'(u_i)}{g(u_i)}.
\]

In other words, particle levels satisfy

\[
\dot{U}_i(t) = aU_i^2(t) - bU_i(t),
\]

and a particle with level \( z \) gives birth at rate \( 2a(r - z) \) to a particle whose initial level is uniformly distributed between \( z \) and \( r \).

\( N(t) = \# \{ i : U_i(t) < r \} \)

\( \alpha_r(n, du) \) the joint distribution of \( n \) iid uniform \([0, r]\) random variables.
A calculation

As before, \( \hat{f}(n) = \int f(u, n) \alpha_r(n, du) = e^{-\lambda g n} \), \( e^{-\lambda g} = \frac{1}{r} \int_0^r g(u) du \)

To calculate \( \int A f(u, n) \alpha_r(n, du) \), observe that

\[
r^{-1} 2a \int_0^r g(z) \int_z^r (g(v) - 1) dv = ar e^{-2\lambda g} - 2ar^{-1} \int_0^r g(z)(r - z) dz
\]

and

\[
r^{-1} \int_0^r (az^2 - bz) g'(z) dz = -r^{-1} \int_0^r (2az - b)(g(z) - 1) dz
\]

\[
= -2ar^{-1} \int_0^r zg(z) dz + ar + b(e^{-\lambda g} - 1).
\]

Then

\[
\int A f(u, n) \alpha_r(n, du) = ne^{-\lambda g(n-1)} (ar e^{-2\lambda g} - 2ar e^{-\lambda g} + ar + b(e^{-\lambda g} - 1))
\]

\[
= \hat{C} \hat{f}(n),
\]

where

\[
\hat{C} \hat{f}(n) = arn( \hat{f}(n + 1) - \hat{f}(n) ) + (ar - b)n( \hat{f}(n - 1) - \hat{f}(n) ).
\]
Markov mappings

**Theorem 4**  $A \subset \overline{C}(E) \times \overline{C}(E)$ a pre-generator with bp-separable graph.

$\mathcal{D}(A)$ closed under multiplication and separating.

$\gamma : E \to E_0$, Borel measurable.

$\alpha$ a transition function from $E_0$ into $E$ satisfying

$$\alpha(y, \gamma^{-1}(y)) = 1$$

Define

$$C = \{ (\int_E f(z)\alpha(\cdot, dz), \int_E Af(z)\alpha(\cdot, dz)) : f \in \mathcal{D}(A) \}.$$ 

Let $\mu_0 \in \mathcal{P}(E_0)$, $\nu_0 = \int \alpha(y, \cdot)\mu_0(dy)$.

If $\tilde{Y}$ is a solution of the MGP for $(C, \mu_0)$, then there exists a solution $Z$ of the MGP

for $(A, \nu_0)$ such that $Y = \gamma \circ Z$ and $\tilde{Y}$ have the same distribution on $M_{E_0}[0, \infty)$.

$$E[f(Z(t))|\mathcal{F}^Y_t] = \int f(z)\alpha(Y(t), dz)$$

(at least for almost every $t$).
Conclusion

Let $\tilde{N}$ be a solution of the martingale problem for

$$ C\hat{f}(n) = arn(\hat{f}(n + 1) - \hat{f}(n)) + (ar - b)n(\hat{f}(n - 1) - \hat{f}(n)), $$

that is, $\tilde{N}$ is a branching process with birth rate $ar$ and death rate $(ar - b)$.

Then there exists a solution $(U_1(t), \ldots, U_{N(t)}(t), N(t))$ of the martingale problem for $A$ such that $N$ has the same distribution as $\tilde{N}$. 
The limit as \( r \to \infty \)

If \( n = O(r) \), then the scaling is correct for the Feller diffusion. \( A \) converges for every \( g \) such that \( 0 \leq g \leq 1 \), \( g(z) = 1 \), \( z \geq u_g \).

\[
f(u) = \prod_i g(u_i)
\]

\[
Af(u) = f(u) \sum_i 2a \int_{u_i}^{u_g} (g(v) - 1)dv + f(u) \sum_i (au_i^2 - bu_i) \frac{g'(u_i)}{g(u_i)}.
\]

If \( nr^{-1} \to y \), then \( \alpha_r(n, du) \to \alpha(y, du) \) where \( \alpha(y, du) \) is the distribution of a Poisson process on \([0, \infty)\) with intensity \( y \).

\[
\hat{f}(y) = \alpha f(y) = \int f(u) \alpha(y, du) = e^{-y} \int_0^\infty (1 - g(z))dz = e^{-y\beta_g}
\]

and

\[
\alpha Af(y) = e^{-y\beta_g} \left( 2ay \int_0^\infty g(z) \int_z^\infty (g(v) - 1)dvdz + y \int_0^\infty (az^2 - bz)g'(z)dz \right)
\]

\[
= e^{-y\beta_g} (ay\beta_g^2 - by\beta_g)
\]

\[
= ay \hat{f}''(y) + by \hat{f}'(y)
\]
Particle representation of Feller diffusion

Let \( \{U_i(0)\} \) be a conditionally Poisson process on \([0, \infty)\) with (conditional) intensity \( Y(0) \). Then, \( \{U_i(t)\} \) is conditionally Poisson with intensity \( Y(t) \),

\[
Y(t) = \lim_{r \to \infty} \frac{1}{r} \# \{ i : U_i(t) \leq r \},
\]

and \( Y \) is a Feller diffusion with generator \( Cf(y) = ayf''(y) + byf'(y) \)

\( \gamma : \mathcal{N}(\mathbb{R}) \rightarrow [0, \infty) \)

\[
\gamma(u) = \lim_{r \to \infty} \frac{1}{r} \# \{ i : u_i \leq r \}
\]

\( \alpha(y, du) \) Poisson process distribution on \([0, \infty)\) with intensity \( y \). \( \alpha(y, \gamma^{-1}(y)) = 1. \)
**Extinction**

Assume $U_1(0) < U_2(0) < \cdots$. Then for all $t$, all levels are above

$$U_1(t) = \frac{U_1(0)e^{-bt}}{1 - \frac{a}{b}U_1(0)(1 - e^{-bt})}$$

Let $\tau = \inf\{t : Y(t) = 0\}$

$$P\{\tau > t\} = P\{U_1(0) < [(1 - e^{-bt})a/b]^{-1}\} = 1 - e^{-yb/[(1-e^{-bt})a]}$$

If $b \leq 0$, conditioning on nonextinction for all $t$ is equivalent to setting $U_1(0) = 0$.

The generator becomes

$$Af(u) = f(u) \sum_i 2a \int_{u_i}^{u_g} (g(v) - 1)dv + f(u) \sum_i (au_i^2 - bu_i) \frac{g'(u_i)}{g(u_i)}$$

$$+ f(u)2a \int_0^{u_g} (g(v) - 1)dv$$

and

$$\alpha Af(y) = ay\hat{f}''(y) + (2a + by)\hat{f}'(y)$$
Branching Markov processes

\[ f(x, u, n) = \prod_{i=1}^{n} g(x_i, u_i), \]  where \( g : E \times [0, \infty) \to (0, 1] \)

As a function of \( x \), \( g \) is in the domain \( \mathcal{D}(B) \) of the generator of a Markov process in \( E \), \( g \) is continuously differentiable in \( u \), and \( g(x, u) = 1 \) for \( u \geq r \).

\[ Af(x, u, n) = f(x, u, n) \sum_{i=1}^{n} \frac{Bg(x_i, u_i)}{g(x_i, u_i)} + f(x, u, n) \sum_{i=1}^{n} 2a(x_i) \int_{u_i}^{r} (g(x_i, v) - 1)dv \]

\[ + f(x, u, n) \sum_{i=1}^{n} \left( a(x_i)u_i^2 - b(x_i)u_i \right) \frac{\partial_u g(x_i, u_i)}{g(x_i, u_i)} \]

Each particle has a location \( X_i(t) \) in \( E \) and a level \( U_i(t) \) in \( [0, r] \).

The locations evolve independently as Markov processes with generator \( B \), the levels satisfy

\[ \dot{U}_i(t) = a(X_i(t))U_i^2(t) - b(X_i(t))U_i(t) \]

and particles that reach level \( r \) die.

Particles give birth at rates \( 2a(X_i(t))(r - U_i(t)) \); the initial location of a new particle is the location of the parent at the time of birth; and the initial level is uniformly distributed on \( [U_i(t), r] \).
Generator for $X(t) = (X_1(t), \ldots, X_N(t))$

Setting $e^{-\lambda_g(x_i)} = r^{-1} \int_0^r g(x_i, z)dz$ and $\hat{f}(x, n) = e^{-\sum_{i=1}^n \lambda_g(x_i)}$, and calculating as in the previous example, we have

$$C\hat{f}(x, n) = \sum_{i=1}^n B_{x_i}\hat{f}(x, n) + \sum_{i=1}^n a(x_i)r(\hat{f}((x, x_i), n+1) - \hat{f}(x, n))$$

$$+ \sum_{i=1}^n (a(x_i)r - b(x_i))(\hat{f}(d(x|x_i), n-1) - \hat{f}(x, n)),$$

where $B_{x_i}$ is the generator $B$ applied to $\hat{f}(x, n)$ as a function of $x_i$. 
Infinite population limit

Letting \( r \to \infty \), \( Af \) becomes

\[
Af(x, u) = f(x, u) \sum_i \frac{B g(x_i, u_i)}{g(x_i, u_i)} + f(x, u) \sum_i 2a(x_i) \int_{u_i}^{u_g} (g(x, v) - 1) dv \\
+ f(x, u) \sum_i (a(x_i) u_i^2 - b(x_i) u_i) \frac{\partial g(x_i, u_i)}{g(x_i, u_i)}
\]

Particle locations evolve as independent Markov processes with generator \( B \).

Levels satisfy

\[
\dot{U}_i(t) = a(X_i(t)) U_i^2(t) - b(X_i(t)) U_i(t)
\]

A particle with level \( U_i(t) \) gives birth to new particles at its location \( X_i(t) \) and initial level in the interval \([U_i(t) + c, U_i(t) + d]\) at rate \( 2a(X_i(t))(d - c) \).

A particle dies when its level hits \( \infty \).
The measure-valued limit

For $\mu \in M_f(E)$, let $\alpha(\mu, dx \times du)$ be the distribution of a Poisson random measure on $E \times [0, \infty)$ with mean measure $\mu \times m$. Then setting $h(y) = \int_0^\infty (1 - g(y, v))dv$

$$\alpha f(\mu) = \int f(x, u)\alpha(\mu, dx \times du) = \exp\{-\int_E h(y)\mu(dy)\},$$

and

$$\alpha Af(\mu) = \exp\{-\int_E h(y)\mu(dy)\} \left[ \int_E \int_0^\infty Bg(y, v) dv \mu(dy) + \int_E \int_0^\infty 2a(y)g(y, z) \int_z^\infty (g(y, v) - 1) dv dz \mu(dy) + \int_E \int_0^\infty (a(y)v^2 - b(y)v) \partial_v g(y, v) dv \mu(dy) \right] = \exp\{-\int_E h(y)\mu(dy)\} \int_E (-Bh(y) + a(y)h(y)^2 - b(y)h(y)) \mu(dy)$$

It follows that the Cox measure (or more precisely, the $E$ marginal of the Cox measure) corresponding to the particle process at time $t$, call it $Z(t)$, is a solution of the martingale problem for $\mathcal{A} = \{(\alpha f, \alpha Af) : f \in \mathcal{D}\}$. 
Branching processes in random environments

$a$ and $b$ functions of another stochastic process $\xi$, say an irreducible finite Markov chain with generator $Q$.

\[ f(l, u, n) = f_0(l)f_1(u) = f_0(l) \prod_{i=1}^{n} g(u_i), \]

\[ A_r f(l, u, n) = r f_1(u) Q f_0(l) + f(l, u, n) \sum_{i=1}^{n} 2a(l) \int_{u_i}^{r} (g(v) - 1) dv \]

\[ + f(l, u, n) \sum_{i=1}^{n} (a(l)u_i^2 - \sqrt{rb(l)}u_i) \frac{g'(u_i)}{g(u_i)}, \]

which projects to

\[ C_r \hat{f}(l, n) = r Q \hat{f}(l, n) + a(l)rn(\hat{f}(l, n + 1) - \hat{f}(n)) \]

\[ +(ra(l) - \sqrt{rb(l)})n(\hat{f}(l, n - 1) - \hat{f}(l, n)), \]

where $\hat{f}(l, n) = f_0(l)e^{-\lambda g^n}$. The process corresponding to $C_r$ is a branching process in a random environment determined by $\xi$. 
Scaling limit

Writing the process corresponding to $A_r$ as

$$(\xi(rt), X_1(t), \ldots, X_{N_r}(t), U_1(t), \ldots, U_{N_r}(t))$$

the process corresponding to $C_r$ is $(\xi(rt), N_r(t))$.

Note that the levels satisfy

$$\dot{U}_i(t) = a(\xi(rt))U_i^2(t) - \sqrt{rb(\xi(rt))}U_i(t).$$

Let $\pi$ be the stationary distribution for $Q$ and assume that $\sum_l \pi(l)b(l) = 0$. Then

$$Z^{(r)}(t) = \sqrt{r} \int_0^t b(\xi(rs)) ds$$

converges to a Brownian motion $Z$ with variance parameter

$$\sum_k \sum_l \pi(k) q_{kl}(h_0(l) - h_0(k))^2 = -2 \sum_l \pi(l)h_0(l)b(l) \equiv 2\bar{c},$$

where $h_0(l)$ is a solution of $Qh_0(l) = b(l)$. In the limit, the levels will satisfy

$$dU_i(t) = (\bar{a}U_i(t)^2 + \bar{c}U_i(t))dt + \sqrt{2\bar{c}}U_i(t)dW(t),$$

(1)

where $\bar{a} = \sum \pi(l)a(l)$.

Note that the limiting levels are all driven by the same Brownian motion.
Limiting generator

\[ h_1(l, u, n) = h_0(l) f_1(u, n) \sum_{i=1}^{n} u_i \frac{g'(u_i)}{g(u_i)}, \]

Passing to the limit as \( r \to \infty \), \( A_r(f_1 + \frac{1}{\sqrt{r}h}) \) converges to

\[ \tilde{A} f_1(u, l) \]

\[ = f_1(u) \sum_i 2a(l) \int_{u_i}^{\infty} (g(v) - 1)dv + f_1(u) \sum_i a(l) u_i^2 \frac{g'(u_i)}{g(u_i)} \]

\[ -h_0(l) b(l) f_1(u) \sum_{j=1}^{n} \left( \sum_{i \neq j} u_j u_i \frac{g'(u_i)g'(u_j)}{g(u_i)g(u_j)} + \frac{u_j g'(u_j) + u_j^2 g''(u_j)}{g(u_j)} \right). \]

An additional perturbation \( h_2 \) gives \( A_r(f_1 + \frac{1}{\sqrt{r}h_1} + \frac{1}{r}h_2) \) converging to

\[ A f_1(u) \]

\[ = f_1(u) \sum_i 2\overline{a} \int_{u_i}^{\infty} (g(v) - 1)dv + f_1(u) \sum_i \overline{a} u_i^2 \frac{g'(u_i)}{g(u_i)} \]

\[ + \overline{c} f_1(u) \sum_{j=1}^{n} \left( \sum_{i \neq j} u_j u_i \frac{g'(u_i)g'(u_j)}{g(u_i)g(u_j)} + \frac{u_j g'(u_j) + u_j^2 g''(u_j)}{g(u_j)} \right). \]
Limiting diffusion

Let

\[ \beta_g = \int_0^\infty (1 - g(z))dz = \int_0^\infty zg'(z)dz = -\frac{1}{2} \int_0^\infty z^2 g''(z)dz. \]

We have

\[ \alpha Af(y) = e^{-y\beta_g} \left( 2\bar{a}y \int_0^\infty g(z) \int_z^\infty (g(v) - 1)dvdz + y \int_0^\infty (\bar{a}z^2 + \bar{c}z)g'(z)dz \right. \]
\[ + \bar{c}y^2 \left( \int_0^\infty zg'(z)dz \right)^2 \left. + \bar{c}y \int_0^\infty z^2 g''(z)dz \right) \]
\[ = e^{-y\beta_g}((\bar{a}y + \bar{c}y^2)\beta_g^2 + \bar{c}y\beta_g) \]
\[ = C \hat{f}(y), \]

where

\[ C \hat{f}(y) = (\bar{a}y + \bar{c}y^2)\hat{f}'''(y) + \bar{c}y\hat{f}'(y), \]

which identifies the diffusion limit for \( r^{-1}N_r \).
Model with local annihilation and replacement

$D$ the location space

$E$ the type space

$\mathcal{N}(S)$ denotes the space of counting measures on $S$

$\beta$ is a $\sigma$-finite measure on $D$

$\xi$ is a Poisson random measure on $[0, \infty) \times D \times [0, \infty)^2$ with mean measure $m \times \beta \times \mu_{h,q}$ such that the point $(s, y, h, q)$ means that at time $s$, a birth/death event occurs in a ball of radius $h$ and center $y$ with thinning probability $1 - \frac{1}{1+q} = \frac{q}{1+q}$.

For each point $(s, y, h, q)$, $\nu_{s,y,h,q}$ is a Poisson random measure on $B_h(y) \times [0, \infty)$ with mean measure $\lambda_q \beta \times m$ ($m$ being Lebesgue measure).
Stochastic equation

$\eta_t$ is the counting measure given by the locations, types, and levels of the particles alive at time $t$.

For $\eta \in \mathcal{N}(D \times E \times [0, \infty))$, let

$$\theta_{y,h}(\eta) \equiv \min(B_h(y), \eta) = \min\{u : (x, v, u) \in \eta, x \in B_h(y)\},$$

and let $\alpha_{y,h}(\eta) \equiv \text{argmin}(B_h(y), \eta)$ denote the value of $v$ in the $(x, v, u) \in \eta$ with $u = \min(B_h(y), \eta)$. $\alpha_{y,h}(\eta)$ will give the type of the parent of the new individuals born in $B_h(y)$.

$f(x, v, u) = 0$ for $u > r$ and $f(x, v, 0) = 0$

$$\int f(x, v, u)\eta_t(dx \times dv \times du)$$

$$= \int f(x, u)\eta_0(dx \times dv \times du)$$

$$+ \int_{[0,t] \times F} \int_{D \times E \times [0,r]} (f(x, v, (1 + q)r \frac{u - \theta_{y,h}(\eta_{s-})}{r - \theta_{y,h}(\eta_{s-})} - f(x, v, u))$$

$$\times 1_{B_h(y)}(x)\eta_{s-}(dx \times dv \times du)\xi(ds \times dy \times dh \times dq)$$

$$+ \int_{[0,t]} 1_{\{\eta_{s-}(B_h(y)) > 0\}} \langle f(\cdot, \alpha_{y,h}(\eta_{s-}), \cdot), \nu_{s,y,h,q} \rangle \xi(ds \times dy \times dh \times dq)$$
Generator for particle model

Let $0 \leq g \leq 1$, and $g(x, v, u) = 1$ for $u \geq r$ and $u = 0$. Let $h(x, v) = \int_0^\infty (1 - g(x, v, u)) du$ and

$$
\gamma_{y,h,q}^r g(x, v, u, \eta) = g(x, v, (1 + q)r \frac{u - \theta_{y,h}(\eta)}{r - \theta_{y,h}(\eta)}).
$$

For $\eta = \sum \delta_{(x_i, v_i, u_i)}$, define $f(\eta) = \prod g(x_i, v_i, u_i)$. Then, observing that

$$
E[\exp\{\int \log g(x, v, u) \nu_{s,y,h,q}(dx \times du)\}] = \exp\{-\lambda q \int_{B_h(y)} h(x, v) \beta(dx)\},
$$

the generator applied to $f$ is

$$
A_r f(\eta) = f(\eta) \int (\exp\{-\lambda q \int_{B_h(y)} h(x, \alpha_{y,h}(\eta)) \beta(dx)\} \prod_{(x,v,u) \in \eta, x \in B_h(y)} \frac{\gamma_{y,h,q}^r g(x,v,u,\eta)}{g(x,v,u)} - 1) \beta(dy).
$$
Projected generator

For $\mu \in \mathcal{N}(D \times E)$, let $\alpha_r(\mu, d\eta)$ be the distribution of the random counting measure on $D \times E \times [0, r]$ obtained by assigning iid uniform-$[0, r]$ levels to each point in $\mu$. Define $\hat{h}$ by

$$e^{-\hat{h}(x,v)} = r^{-1} \int_0^r g(x, v, u)du = 1 - r^{-1}h(x, v),$$

and observe that

$$\frac{1}{1 + q}e^{-\hat{h}(x,v)} + \frac{q}{1 + q} = 1 - \frac{1}{r(1 + q)}h(x, v) \quad (3)$$

$$\alpha f(\mu) \equiv \int f(\eta)\alpha(\mu, d\eta) = e^{-\int_{D \times E} \hat{h}d\mu}$$
\[ \alpha A_r f(\mu) = \int \left( \frac{1}{\mu(B_h(y) \times E)} \right) \int_{B_h(y) \times E} \frac{\exp\{-\lambda_q \int_{B_h(y)} h(x, v) \beta(dx)\}}{1 - \frac{1}{r(1+q)} h(x, v)} \mu(dz \times dv) \]

\[ \times \exp\{-\int_{(D-B_h(y)) \times E} \hat{h}d\mu\} \]

\[ \times \left( \prod_{(x', v') \in \mu, x' \in B_h(y)} (1 - \frac{1}{r(1+q)} h(x', v')) \right) \]

\[ -\hat{f}(\mu) \beta(dy) \mu_{h,q}(dh, dq) \]

Writing the first expression in the integrand as \( I_1 \times I_2 \times I_3 \), the normalized integral, \( I_1 \), is understood to be 1 if \( \mu(B_h(y) \times E) = 0 \). The denominator of the integrand in \( I_1 \) cancels the corresponding factor in \( I_3 \) and reflects that fact that each parent is removed from the population. The numerator is just the moment generating function of the offspring distribution. With reference to the left side of (3), \( I_3 \) reflects the fact that each non-parent particle already in \( B_h(y) \) at the time of the birth event dies with probability \( \frac{q}{1+q} \).
Large system limit

Passing to the limit $r \to \infty$, the stochastic equation becomes

$$\int f(x, v, u) \eta_t(dx \times dv \times du) = \int f(x, v, u) \eta_0(dx \times dv \times du)$$

$$+ \int_{[0,t] \times F} \int_{D \times E \times [0,\infty)} (f(x, v, (1 + q)(u - \theta_{y,h}(\eta_{s-}))) - f(x, v, u))$$

$$\times 1_{B_h(y)}(x) \eta_{s-}(dx \times dv \times du) \xi(ds \times dy \times dh \times dq)$$

$$+ \int_{[0,t]} 1_{\{\eta_{s-}(B_h(y)) > 0\}} \langle f(\cdot, \alpha_{y,h}(\eta_{s-}), \cdot), \nu_{s,y,h,q} \rangle \xi(ds \times dy \times dh \times dq).$$
Computation of generator

For $g \in B(D \times E \times [0, \infty)$ and $\eta \in \mathcal{N}(D \times E \times [0, \infty)$, the limit of $\gamma_{y,h,q}^r g$ is

$$\gamma_{y,h,q}^r g(x, v, u, \eta) = g(x, v, (1 + q)(u - \theta_{y,h}(\eta))).$$

If $0 \leq g \leq 1$, and $g(x, v, u) = 1$ for $u \geq u_g$ and $u = 0$, then for $r > u_g$, $f(\eta) = \prod g(x_i, v_i, u_i)$ is in the domain of $A_r$ and $A_r f(\eta)$ converges to

$$Af(\eta) = f(\eta) \int \left( \exp\{ -\lambda_q \int_{B_h(y)} h(x, \alpha_{y,h}(\eta)) \beta(dx) \} \right) \times \prod_{(x, v, u) \in \eta, x \in B_h(y)} \frac{\gamma_{y,h,q} g(x, v, u, \eta)}{g(x, v, u)} - 1 \bigg) \beta(dy) \mu_{h,q}(dh, dq)$$
Projected generator

$\alpha(\mu, d\eta)$ distribution of Poisson random measure with mean measure $\mu \times m$ on $D \times E \times [0, \infty)$.

$$\hat{f}(\mu) = \alpha f(\mu) = \exp\{-\int_{D \times E} h d\mu\}$$

and

$$\alpha Af(\mu) = \int \left( \frac{1}{\mu(B_h(y) \times E)} \right) \int_{B_h(y) \times E} \exp\{-\lambda_q \int_{B_h(y)} h(x, v) \beta(dx)\} \mu(dz \times dv)$$

$$\times \exp\{-\int_{(D-B_h(y)) \times E} h d\mu\} \exp\{-1 + q\}^{-1} \int_{B_h(y)} h d\mu\}$$

$$- \hat{f}(\mu) \beta(dy) \mu_{h, q}(dh, dq).$$
Exit measures

Let $B$ be the generator of a diffusion and $D$ a bounded domain with all points in $\partial D$ regular.

\[
Af(x, u) = f(x, u) \sum_i 1_D(x_i) \frac{Bg(x_i, u_i)}{g(x_i, u_i)} \\
+ f(x, u) \sum_i 1_D(x_i) 2a(x_i) \int_{u_i}^{u_g} (g(x_i, v) - 1) dv \\
+ f(x, u) \sum_i 1_D(x_i)(a(x_i)u_i^2 - b(x_i)u_i) \frac{\partial u_i g(x_i, u_i)}{g(x_i, u_i)}
\]

For simplicity assume $\inf_x a(x) > 0, b \leq 0$, then $\tau = \inf\{t : \sum_i 1_D(X_i(t) > 0\} < \infty$. Define $\xi = \sum_i \delta(X_i(\tau), U_i(\tau))$ and

\[
Z_D(\Gamma) = \lim_{r \to \infty} \frac{1}{r} \sum_{i: U_i(\tau) \leq r} 1_{\Gamma}(X_i(\tau))
\]

the exit measure.
Nonlinear PDE

Assume $\{(X_i(0), U_i(0))\}$ is a Poisson random measure with mean measure $\delta_x \times m$. Then for $h \geq 0$, bounded and continuous,

$$u(x) = -\log E[e^{-\langle h, Z^x_D \rangle}]$$

solves the differential equation

$$Bu = au^2 - bu$$
$$u = h \text{ on } \partial D.$$

$$e^{-\langle h, Z^x_D \rangle} = E[e^{\int \log(1-h/r)1_{[0,r]}d\xi} | Z_D] = E[e^{\sum_{i(\tau) \leq r} \log(1-h(X_i(\tau))/r)} | Z_D].$$

Taking expectations,

$$E[e^{-\langle h, Z^x_D \rangle}] = E\left[ \prod_{U_i(\tau) \leq r} \left( 1 - \frac{h(X_i(\tau))}{r} \right) \right]$$
Homogenization

\[ Bu(x) = a(x, x/\epsilon)u(x)^2 - b(x, x/\epsilon)u(x) \]
\[ u = h \text{ on } \partial D \]

\( a(x, y), b(x, y) \) periodic in \( y \), and

\[ Bg(x) = \frac{1}{2} \sum_{j,k=1}^{d} c_{jk}(x) \frac{\partial^2 g}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{d} d_j(x) \frac{\partial g}{\partial x_j}(x). \]

Particle generator:

\[ A^\epsilon f(x, u) = f(x, u) \sum_i \frac{Bg(x_i, u_i)}{g(x_i, u_i)} 1_D(x_i) \]
\[ + f(x, u) \sum_i 2a(x_i, x_i/\epsilon) \int_{u_i}^{u_g} (g(x_i, v) - 1) dv 1_D(x_i) \]
\[ + f(x, u) \sum_i (a(x_i, x_i/\epsilon)u_i^2 - b(x_i, x_i/\epsilon)u_i) \frac{\partial u_i g(x_i, u_i)}{g(x_i, u_i)} 1_D(x_i) \]
Effective coefficients

Let $\bar{a}(x) = \int a(x, y)\pi_x(dy)$, $\bar{b}(x) = \int b(x, y)\pi_x(dy)$, where $\pi_x$ is determined by

$$\int \hat{B}g(x, y)\pi_x(dy) = 0, \quad g \in \mathbb{R}^d,$$

where

$$\hat{B}g(x, y) = \frac{1}{2} \sum_{j,k=1}^d c_{jk}(x) \frac{\partial^2 g}{\partial y_j \partial y_k}(y).$$

$$\lim_{\epsilon \to 0} E[e^{-\langle Z_\epsilon D, h \rangle}] = \lim_{\epsilon \to 0} E\left[ \prod_{U_i^\epsilon(\tau) \leq r} \left( 1 - h(X_i(\tau)) \right) \right] = E[e^{-\langle Z D, h \rangle}]$$
Conditioning on non-extinction

Let $a$ be constant and $b \equiv 0$. and let $U_*(0)$ be the minimum of the initial levels.

\[ U_*(t) = \frac{U_*(0)}{1 - aU_*(0)t} \]

Let $\tau$ be the time of extinction, Then $\{\tau > T\} = \{U_*(0) < \frac{1}{aT}\}$. Conditioning on $\{\tau > T\}$ and letting $T \to \infty$ is equivalent to conditioning on the initial Poisson process having a level at zero. The resulting generator becomes

\[
Af(x, u) = f(x, u) \sum_i \frac{B g(x_i, u_i)}{g(x_i, u_i)} + f(x, u) \sum_{i>0} 2a \int_{u_i}^{r_g} (g(x_i, v) - 1) dv + f(x, u) \sum_{i>0} au_i^2 \frac{\partial u_i g(x_i, u_i)}{g(x_i, u_i)},
\]
Generator for measure-valued process

The generator for the measure-valued process is given by setting

\[ \alpha_0 f(\mu) = \int f(x, u)\alpha_0(\mu, dx \times du) = \frac{1}{|\mu|} \int_E g(z, 0)\mu(dz) \exp\{-\langle h, \mu \rangle\}, \]

and

\[ \alpha_0 Af(\mu) = \langle -Bh(y) + ah(y)^2, \mu \rangle \frac{1}{|\mu|} \int_E g(z, 0)\mu(dz) \exp\{-\langle h, \mu \rangle\} \]

\[ + \frac{1}{|\mu|} \int_E (Bg(z, 0) - 2ag(z, 0)h(z))\mu(dz) \exp\{-\langle h, \mu \rangle\} \]
Uniqueness

Corollary 5 If uniqueness holds for the MGP for $(A, \nu_0)$, then uniqueness holds for the $M_{E_0}[0, \infty)$-MGP for $(C, \mu_0)$. If $\tilde{Y}$ has sample paths in $D_{E_0}[0, \infty)$, then uniqueness holds for the $D_{E_0}[0, \infty)$-martingale problem for $(C, \mu_0)$. 
Abstract

Poisson representations of measure-valued processes

Measure-valued diffusions and measure-valued solutions of stochastic partial differential equations can be represented in terms of the Cox measures of particle systems that are conditionally Poisson at each time $t$. The representations are useful for characterizing the processes, establishing limit theorems, and analyzing the behavior of the measure-valued processes. Examples will be given and some of the useful methodology will be described.