Problem set 2: Finite Element Method, basics, elliptic equation

1. show your derivation; 2. late homework not accepted

Due Apr-1, Friday 5:00pm.

Neumann boundary condition

Consider the model problem with Neumann boundary conditions at both ends:

\[
\begin{aligned}
&u''(x) = f(x), \quad 0 < x < 1, \\
&u'(0) = \sigma_0, \quad u'(1) = \sigma_1.
\end{aligned}
\] (1)

(a) By integrating (1) directly, show that problem has a solution if and only if it is compatible:

\[
\int_0^1 f(x)dx = \sigma_1 - \sigma_0
\] (2)

Furthermore, show that the solution, when it exists, is not unique.

(b) Derive a second-order centered finite difference discretization of problem (1):

- Write the scheme in matrix form

\[
A\vec{u} = f
\] (3)

with A begin symmetric and tridiagonal. Show that A is singular and find non-trivial zero (find \vec{v} such that A\vec{v} = 0).

- Give a solvability condition for (3) and compare it with the compatibility condition.

(c) Solve the linear system (3) assuming the solvability condition is satisfied. How many dimension is the solution space?

Variational Formulation

We study the variational form of the elliptic equation with variable coefficient:

\[
- \nabla \cdot (k(x)\nabla u) + u = f, \quad x \in \Omega
\] (4)

For simplicity, we assume:

- \( \Omega = \Omega_1 \cup \Omega_2 \) and they share common boundary S.

- the variable coefficient is piecewise constant: \( k(x) = \kappa_i, \quad x \in \Omega_i, \quad i = 1, 2. \)
zero boundary condition for the $\Omega_1$ side and flux boundary condition for $\Omega_2$ side:

$$u = 0, \quad x \in \Gamma_1, \quad -\nabla_x u \cdot n = g \quad x \in \Gamma_2.$$  \hspace{1cm} (5)

Here $n$ is the unit vector pointing out of the domain.

(a) Show it is equivalent to solving the following PDE system:

$$\begin{cases}
-\kappa_j \Delta u_j + u_j = f, & \text{in } \Omega_j \\
u_1 = 0, & \text{on } \Gamma_1 \\
-\nabla_x u_2 \cdot n = g, & \text{on } \Gamma_2 \\
\kappa_1 \frac{\partial u_1}{\partial n} = \kappa_2 \frac{\partial u_2}{\partial n} & \text{on } S
\end{cases}$$  \hspace{1cm} (6)

$n$ here is unit vector pointing from one $\Omega_i$ to another. The direction does not matter. $u_j$ is $u$ restricted on $\Omega_j$.

(b) Find the corresponding variational problem, and show that the bilinear operator you obtain is symmetric, continuous, coercive and linear operator is also continuous, indicating one solution exists.

(c) Find the corresponding minimization problem.

**Euler-Bernoulli equation**

Consider the Euler-Bernoulli equation

$$\frac{d^4 u}{dx^4} = f(x), \quad 0 < x < 1.$$  \hspace{1cm} (7)

It is used to describe the deflection $u$ of a clamped beam subject to a transversal force with intensity $f$.

(a) Show the equivalent variational form would be to find $u$ such that:

$$(u'', v'') = (f, v), \forall v \in V,$$  \hspace{1cm} (8)

where

$$V = \{ v : v \in C_1[0, 1], v(0) = v'(0) = v(1) = v'(1) = 0, v'' \text{ piecewise continuous and bounded} \}.$$  

(b) For $I = [a, b]$ an interval, define $P_3(I) = \{ v : v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad x \in I \}$. Show that $v \in P_3(I)$ is uniquely determined by the values $v(a), v'(a), v(b)$ and $v'(b)$. Find the corresponding local basis functions. (Hint: count the number of degree of freedom and write use the values to fix the coefficients.)

(c) Construct a finite-dimensional subspace $V_h$ consisting piecewise cubic polynomials on the mesh $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Find suitable representations for the functions in $V_h$. 

(d) Formulate a finite element method for on space $V_h$. Find out explicit expression for each entry of the stiff matrix and load vector.

(e) Derive the error estimate
\[
\|(u - u_h)''\|_2 \leq ||(u - v)''\|_2, \forall v \in V_h.
\] (9)

You are given the estimate that cubic Hermite interpolant of $u$, denoted as $I_h u \in V_h$, satisfies the following:
\[
\|u''(x) - (I_h u)''(x)\| \leq \frac{1}{h^2} \max_{0 \leq \xi \leq 1} |u^{(4)}(\xi)|,
\] (10)

show that
\[
\|(u - u_h)''\| \leq Ch^2 \max_{0 \leq \xi \leq 1} |u^{(4)}(\xi)|
\] (11)

(f) Write a computer program to solve
\[
\begin{aligned}
\frac{d^4}{dx^4} u &= g(x), \\
\frac{d}{dx} u(0) &= u'(0) = u(1) = u'(1) = 0
\end{aligned}
\] (12)

If we use $g(x) = \frac{d^4}{dx^4} e^x x^2 (1 - x)^2 = e^x (x^4 + 14x^3 + 49x^2 + 32x - 12)$, the exact solution is $u(x) = e^x x^2 (1 - x)^2$.

- give a brief description of your algorithm, in particular, the method you use to evaluate the load vector $b$ (choose your favorite numerical integral method, but make sure the error here is not too big, and the error from $A$ still dominates).
- tabulate the max-norm errors $e_n = \max_{x_j} |u_h(x_j) - u(x_j)|$ and show the numerical convergence order by performing linear regression of $\log(e_n)$ v.s $\log(n)$.
- plot your finite element solution $u_h$ along with the real solution.
- attach your code.