Math 521 Homework

November 24, 2009

- Homework 1

  1. Determine which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are $1-1$, onto, neither, or both:

     (a) $f(x) = cos\ x$

     (b) $f(x) = e^x$

     (c) $f(x) = x^3$

     (d) $f(x) = x^3 - x^2$

  2. Which of the following formulas are true and which are false? Give a counterexample to the false ones.

     (a) $F(A \cup B) = F(A) \cup F(B)$

     (b) $F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$

     (c) $F(A \cap B) = F(A) \cap F(B)$

     (d) $F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$

- Homework 2: Show
  1. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$
  2. $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\} \sim [0, 1)$
  3. $\mathbb{R} \sim (\mathbb{R}^+ \setminus \{0\})$
  4. * (* means challenging!) $\mathbb{R}^+ \sim (\mathbb{R}^+ \setminus \{0\})$
- Homework 3
  1. Prove for each $n \in \mathbb{N}$, the set of polynomials of degree n with integer coefficients is countable.
  2. Prove the set of polynomials with integer coefficients is countable.
  3. * Let S be an infinite set and $T : S \to S$. Prove there exists a proper subset, A, of S such that $T(A) \subset A$.
  4. Find a counterexample to the conclusion of 3 when S is finite. (I.e. find a T for which $T(S) = S$ and no proper subset has this property).
- Homework 4: P. 276-77, # 6, 9, 10, 13
- Homework 5
  1. Indicate whether the following sequences converge or diverge. Find the limit if the sequence converges.
     (a) $(-1)^n(1 - (1/n))$
     (b) $(n^2 + n + 1)/(7n^2 + 14n + 2)$
     (c) P. 35, # 16(a-c)
  2. P. 281, # 5
  3. Prove the Comparison Test: If the sequences $(a_m), (b_m), (c_m)$ satisfy $a_m \leq b_m \leq c_m$ and $\lim_{m\to\infty} a_m = L = \lim_{m\to\infty} c_m$, then $\lim_{m\to\infty} b_m = L$.

- Homework 6

  1. P. 35, # 18

  2. P. 43. # 10, 11(a-b)

- Homework 7

  1. Suppose $(a_m) \subset \mathbb{R}$ (or $\mathbb{R}^n$)) and $f : \mathbb{N} \to \mathbb{N}$ is a $1-1$ onto map. Then $(a_{f(m)})$ is called a rearrangement of (a_m). Prove if $\lim_{m\to\infty} a_m = L$, then $\lim_{m\to\infty} a_{f(m)} = L$.

  2. If $\liminf_{n\to\infty} x_n = L = \limsup_{n\to\infty} x_n$, then $L = lim_{n\to\infty} x_n$

- Homework 8

  1. Find the points of accumulation of $S = \{\pm 1/n \mid n \in \mathbb{N}\}$.
  2. Is there a set $S \subset \mathbb{R}^n$ whose set of points of accumulation $= B_1(0)$?

- Homework 9

  1. P.57 #1 (a-d), 2, 6, 9

  2. Give an example of a function, $f : \mathbb{R} \to \mathbb{R}$ with f continuous only on $\mathbb{N}$.

  3. * Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) = 1/q$ if $x = p/q \in \mathbb{Q}$ where p, q are in lowest terms. Show f is continuous at irrational points and discontinuous at rational points.

- Homework 10A: P.288 # 2(a)-(d), 7

- Homework 10B

  1. P.288 #10-12

2. What is $\partial\mathbb{N}, int\ \mathbb{N}, cl\ \mathbb{N}$?

- Homework 11
  1. P. 46 # 2
  2. Give an example of a sequence which has subsequences converging to
     (a) each member of $\mathbb{N}$
     (b) each member of $\mathbb{Q}^+$
     (c) each member of $\mathbb{R}^+$
- Homework 12
  1. For $p > 1$, show $\sum 1/n^p < \infty$
  2. For $0 < p \leq 1$, show $\sum 1/n^p = \infty$
- Homework 13: P. 52 # 1(a-e), 4 (a-c), 7, P.304 # 8
- Homework 14
  1. Give an example of $f \in C(K, \mathbb{R}),\ K \subset \mathbb{R}$ which need not have a maximum if
     (a) K is bounded
     (b) K is closed
  2. Prove $K \subset \mathbb{R}$ is compact iff every $f \in C(K, \mathbb{R})$ has a maximum
  3. If $(x_j) \subset \mathbb{R}^n$, prove x_j converges to x as $j \to \infty$ iff everysubsequence of (x_j) has a subsequence that converges to x.

- Homework 15

    1. Show $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

    2. Suppose $f : A \to \mathbb{R}^m,\ A \subset \mathbb{R}^n$. F is called an extension of f to $\bar{A}$ if $F : \bar{A} \to \mathbb{R}^m$ and $F(x) = f(x)$ for $x \in A$. Prove if $f \in C(A, \mathbb{R}^m)$ and f is uniformly continuous on A, f has a continuous extension, F, to $\bar{A}$.

- Homework 16: P. 65 # 4,5,6,7,   P. 309 # 2

- Homework 17: P. 108-109 # 5,6,7,10,15

- Homework 18

    1. P. 109 # 22,23
    2. L'Hospital's Rule: Suppose $f, g \in C([a, b]), z \in (a, b)$, and $f(z) = g(z) = 0$. If f and g are continuously differentiable near z and $g'(z) \neq 0$, $lim_{x \to z} f(x)/g(x) = f'(z)/g'(z)$.

- Homework 19

    1. If f is bounded and integrable,and $\alpha \in \mathbb{R}$, prove αf is integrable and $\int_a^b \alpha f = \alpha \int_a^b f$.
    2. P. 150 # 12; P. 159 #1 (a-c); P. 160 #8.
    3. If f is continuous and $T - periodic$, i.e. $f(x + T) = f(x)$ for all $x \in \mathbb{R}$, then $\int_0^T f = \int_a^{a+T} f$ for all $a \in \mathbb{R}$.

- Homework 20

    1. If f is integrable and $|f| \leq M$, then $\int_a^b |f| \leq M(b - a)$.
    2. P.160 # 9

- Homework 21a: P172 # 1, (b)-(c), 3, 5

- Homework 21b

    1. P. 172 # 1(d), 8
    2. Give a new proof of problem 3 in Homework 19. Hint: consider $\int_a^{a+T} f$ as a function of a and differentiate.

- Homework 22

  1. P.248 # 1,3,4,7
  2. If $f_n(x) = nx^n(1-x)$ on $[0,1]$, show f_n convergers pointwise but not uniformly on $[0,1]$.
  3. Suppose $S \subset \mathbb{R}$, $f_n : S \to \mathbb{R}$ for all $n \in \mathbb{N}$, and $f_n \to f$ uniformly on S. Give a counterexample to the following two statements:
     (a) $f_n^2 \to f^2$ uniformly on S.
     (b) If $f_n \neq 0$ for all $n \in \mathbb{N}$, $1/f_n \to 1/f$ uniformly on S.

- Homework 23

  1. Recall if $A \subset S \subset \mathbb{R}, A$ is dense in S if $S = \bar{A} \bigcap S$. Prove $\mathbb{Q}$ is dense in $\mathbb{R}$.
  2. Suppose $S \subset \mathbb{R}$ and $f_n,\ f \in C(S)$ for all $n \in \mathbb{N}$. If $f_n \to f$ uniformly on A which is dense in S, then $f_n \to f$ uniformly on S.
  3. If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $f_n(x) = f(x + 1/n) \to f(x)$ as $n \to \infty$ uniformly on $\mathbb{R}$.
  4. Recall g is Lipschitz continuous on $D \subset \mathbb{R}^n$ if there is a constant K such that $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \in D$. K is called the Lipschitz constant of g on D. If $f_n \to f$ pointwise in D, with f_n Lipschitz continuous with Lipschitz constant,K for all $n \in \mathbb{N}$, show f is also Lipschitz continuous with Lipschitz constant, K.
  5. * In the setting of problem #4, show if D is compact, $f_n \to f$ uniformly on D.

- Homework 24: P.239 # 1(a), (b), (d), (e), (f).
  Hint for (e): read Corollary 9.11

- Homework 25

  Do the following series converge or diverge? Justify your answer.

  1. $\sum_0^\infty (-1)^n/2^n$
  2. $\sum_1^\infty (-1)^n/n^{1/10}$
  3. $\sum_1^\infty (-1)^n/(1 + 1/n)$

- Homework 26: P.262-3 # 1 (a), (c), 13

- Homework 27: P.262-3 # 2, 3, 8
- Homework 28

  Show $log\ x \to -\infty$ as $x \to 0$ and $log\ x \to \infty$ as $x \to \infty$
- Homework 29
  1. P.123 # 2,3,5,9
  2. * Prove (b) of #9 under the assumption that h is merely continuous.
  3. If f is differentiable and satisfies $f(xy) = f(x) + f(y)$ for all $x, y > 0$, prove $f(x) = c\ log\ x$ for some $c \in \mathbb{R}$.
- Homework 30: P131-132 # 5, 7, 14, *9-10
- Homework 31
  1. Show $(1 - x^2)^n \geq 1 - nx^2$ for $x \in [0, 1]$.
  2. Show for $a \in (0, 1)$, $\sqrt{n}a^n \to 0$ as $n \to \infty$.
  3. P.227, #5.
- Homework 32
  1. Sketch the unit ball in $\mathbb{R}^2$ under

     (a) the usual Euclidean metric,

     (b) $d(x, y) = \sum_1^2 |x_i - y_i|$,

     (c) $d(x, y) = max_{i=1,2}|x_i - y_i|$.
  2. Are the following metrics on $C([0, 1]), \mathbb{R})$? If not, why not?

     (a) $d(f, g) = |f(0) - g(0)|$,

(b) $d(f,g) = \int_0^1 |f-g|dx$,

(c) $d(f,g) = \int_0^1 |f-g|^2 dx$.

- Homework 33

  1. P.321 # 1, 3, 9, P.341 # 6,7
  2. Suppose $A \subset \mathcal{M}$, a metric space. Prove A is closed iff it contains its points of accumulation.

- Homework 34: P. 346-7 # 2, 6, 9a, 10 a,b,c

- Homework 35

  1. If $\mathcal{F} \subset C([a,b],\mathbb{R})$ and $\mathcal{F}$ is equicontinuous, then $\overline{\mathcal{F}}$ is equicontinuous.
  2. Suppose $\mathcal{F}$ is

     (a) not closed but is bounded and equicontinuous

     (b) is not bounded but is closed and equicontinuous

     (c) is not equicontinuous but is closed and bounded

     For each of (a), (b),(c), give a counterexample showing that the conclusion of the Arzela-Ascoli Theorem does not hold.