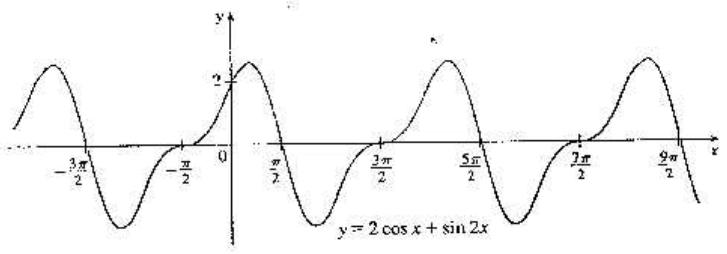
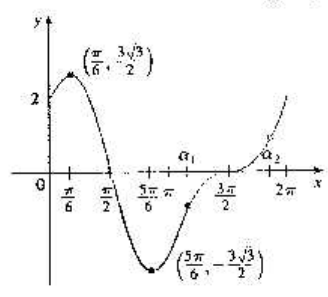


As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$



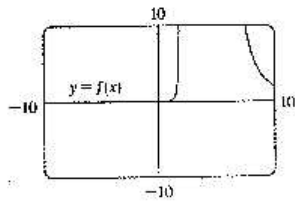


FIGURE 10

EXAMPLE 3 Graph the function $f(x) = \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

SOLUTION Drawing on our experience with a rational function in Example 2, let's start by graphing f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some finer detail and also to zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x-2)^2$ and $(x-4)^4$ in the denominator we expect $x=2$ and $x=4$ to be the vertical asymptotes. Indeed

$$\lim_{x \rightarrow 2} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty \quad \text{and} \quad \lim_{x \rightarrow 4} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty$$

To find the horizontal asymptotes we divide numerator and denominator by x^6 :

$$\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \frac{\frac{1}{x} \left(1 + \frac{1}{x}\right)^3}{\left(1 - \frac{2}{x}\right)^2 \left(1 - \frac{4}{x}\right)^4} \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty$$

so the x -axis is the horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x -intercepts using an analysis like that in Example 9 in Section 1.6. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph doesn't cross the x -axis at 0. But, because of the factor $(x+1)^3$, the graph does cross the x -axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

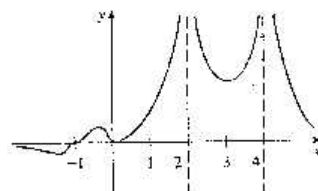


FIGURE 11

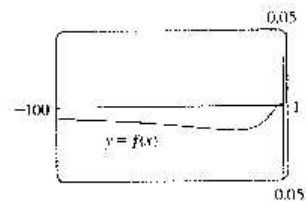


FIGURE 12

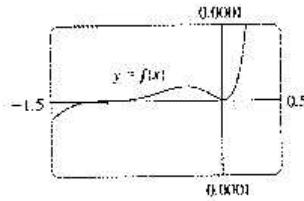


FIGURE 13

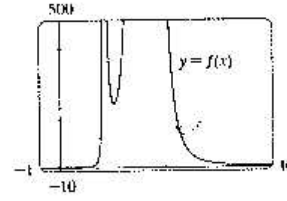
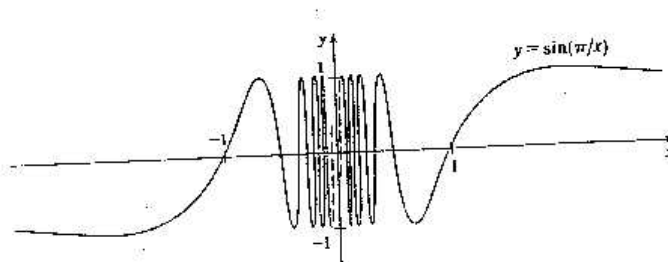


FIGURE 14



EXAMPLE 5 How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

SOLUTION The graphs in Figures 19 and 20 (the special cases $c = 2$ and $c = -2$) show two very different-looking curves. Before drawing any more graphs, let's see what members of this family have in common. Since

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^2 + 2x + c} = 0$$

for any value of c , they all have the x -axis as a horizontal asymptote. A vertical asymptote will occur when $x^2 + 2x + c = 0$. Solving this quadratic equation, we get $x = -1 \pm \sqrt{1 - c}$. When $c > 1$, there is no vertical asymptote (as in Figure 19). When $c = 1$ the graph has a single vertical asymptote $x = -1$ because

$$\lim_{x \rightarrow -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \rightarrow -1} \frac{1}{(x + 1)^2} = \infty$$

When $c < 1$ there are two vertical asymptotes: $x = -1 + \sqrt{1 - c}$ and $x = -1 - \sqrt{1 - c}$ (as in Figure 20).

Now we compute the derivative:

$$f'(x) = -\frac{2x + 2}{(x^2 + 2x + c)^2}$$

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geq 1$ this means that f increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = 1/(c - 1)$. For $c < 1$, $f(-1) = 1/(c - 1)$ is a local maximum value and the intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed in the viewing rectangle $[-5, 4]$ by $[-2, 2]$.

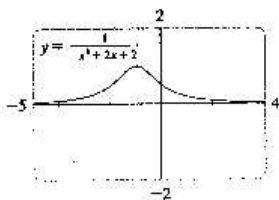


FIGURE 19
 $c = 2$

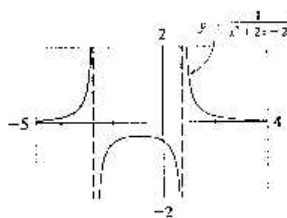


FIGURE 20
 $c = -2$

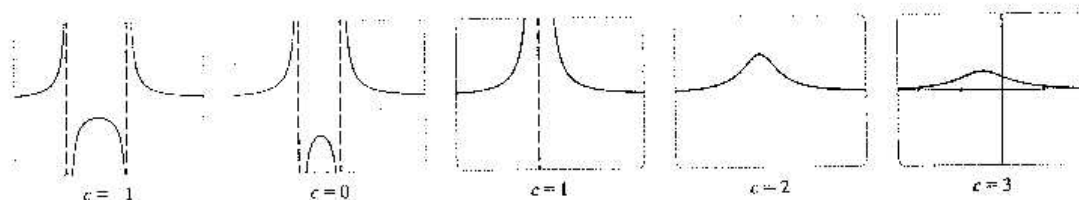


FIGURE 21
The family of functions
 $f(x) = \frac{1}{x^2 + 2x + c}$

As predicted, $c = 1$ is the value at which a transition takes place from two vertical asymptotes to one, and then to none. As c increases from 1, we see that the maximum point becomes lower; this is explained by the fact that $1/(c - 1) \rightarrow 0$ as $c \rightarrow \infty$. As c decreases from 1, the vertical asymptotes become more widely separated because the distance between them is $2\sqrt{1 - c}$, which becomes large as $c \rightarrow -\infty$. Again the maximum point approaches the x -axis because $1/(c - 1) \rightarrow 0$ as $c \rightarrow -\infty$.