

FIGURE 10

EXAMPLE 3 Graph the function $f(x) = \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4}$.

SOLUTION Drawing on our experience with a rational function in Example 2, let's start by graphing f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some finer detail and also to zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x-2)^2$ and $(x-4)^4$ in the denominator we expect $x=2$ and $x=4$ to be the vertical asymptotes. Indeed

$$\lim_{x \rightarrow 2} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty \quad \text{and} \quad \lim_{x \rightarrow 4} \frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \infty$$

To find the horizontal asymptotes we divide numerator and denominator by x^6 :

$$\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \frac{\frac{1}{x} \left(1 + \frac{1}{x}\right)^3}{\left(1 - \frac{2}{x}\right)^2 \left(1 - \frac{4}{x}\right)^4} \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty$$

so the x -axis is the horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x -intercepts using an analysis like that in Example 9 in Section 1.6. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph doesn't cross the x -axis at 0. But, because of the factor $(x+1)^3$, the graph does cross the x -axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

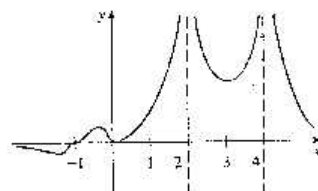


FIGURE 11

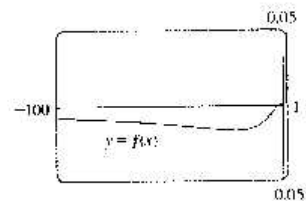


FIGURE 12

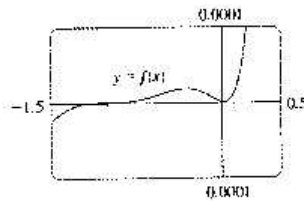


FIGURE 13

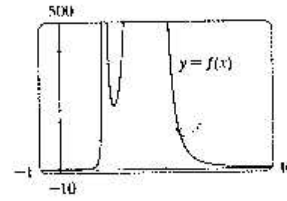


FIGURE 14