

Combinatorics in affine flag varieties

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Dedicated to Gus Lehrer on the occasion of his 60th birthday

Abstract

The Littelmann path model gives a realisation of the crystals of integrable representations of symmetrizable Kac-Moody Lie algebras. Recent work of Gaussent-Littelmann [GL] and others [BG] [GR] has demonstrated a connection between this model and the geometry of the loop Grassmanian. The alcove walk model is a version of the path model which is intimately connected to the combinatorics of the affine Hecke algebra. In this paper we define a refined alcove walk model which encodes the points of the affine flag variety. We show that this combinatorial indexing naturally indexes the cells in generalized Mirkovic-Vilonen intersections.

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1 Introduction

A *Chevalley group* is a group in which row reduction works. This means that it is a group with a special set of generators (the “elementary matrices”) and relations which are generalisations of the usual row reduction operations. One way to efficiently encode these generators and relations is with a Kac-Moody Lie algebra \mathfrak{g} . From the data of the Kac-Moody Lie algebra and a choice of a commutative ring or field \mathbb{F} the group $G(\mathbb{F})$ is built by generators and relations following Chevalley-Steinberg-Tits.

Of particular interest is the case where \mathbb{F} is the field of fractions of \mathfrak{o} , the discrete valuation ring \mathfrak{o} is the ring of integers in \mathbb{F} , \mathfrak{p} is the unique maximal ideal in \mathfrak{o} and $k = \mathfrak{o}/\mathfrak{p}$ is the residue field. The favourite examples are

$$\begin{array}{lll} \mathbb{F} = \mathbb{C}((t)) & \mathfrak{o} = \mathbb{C}[[t]] & k = \mathbb{C}, \\ \mathbb{F} = \mathbb{Q}_p & \mathfrak{o} = \mathbb{Z}_p & k = \mathbb{F}_p, \\ \mathbb{F} = \mathbb{F}_q((t)) & \mathfrak{o} = \mathbb{F}_q[[t]] & k = \mathbb{F}_q, \end{array}$$

where \mathbb{Q}_p is the field of p -adic numbers, \mathbb{Z}_p is the ring of p -adic integers, and \mathbb{F}_q is the finite field with q elements. For clarity of presentation we shall work in the first case where $\mathbb{F} = \mathbb{C}((t))$. The diagram

$$\begin{array}{ccc} & & G = G(\mathbb{C}((t))) \\ & & \cup \quad \cup \\ \mathbb{F} & & \\ \cup & \text{gives} & K = G(\mathbb{C}[[t]]) \xrightarrow{\text{ev}_{t=0}} G(\mathbb{C}) \quad (1.1) \\ \mathfrak{o} \xrightarrow{\text{ev}_{t=0}} k = \mathfrak{o}/\mathfrak{p} & & \cup \quad \cup \quad \cup \\ & & I = \text{ev}_{t=0}^{-1}(B(\mathbb{C})) \xrightarrow{\text{ev}_{t=0}} B(\mathbb{C}) \end{array}$$

where $B(\mathbb{C})$ is the “Borel subgroup” of “upper triangular matrices” in $G(\mathbb{C})$. The *loop group* is $G = G(\mathbb{C}((t)))$, I is the standard *Iwahori subgroup* of G ,

$$G(\mathbb{C})/B(\mathbb{C}) \text{ is the } \textit{flag variety}, \quad (1.2)$$

G/I is the *affine flag variety*, and G/K is the *loop Grassmanian*.

The primary tool for the study of these varieties (ind-schemes) are the following “classical” double coset decompositions, see [St, Ch. 8] and [Mac1, §(2.6)]

Theorem 1.1. *Let W be the Weyl group of $G(\mathbb{C})$, $\widetilde{W} = W \rtimes \mathfrak{h}_{\mathbb{Z}}$ the affine Weyl group, and U^- the subgroup of “unipotent lower triangular” matrices in $G(\mathbb{F})$ and $\mathfrak{h}_{\mathbb{Z}}^+$ the set of dominant elements of $\mathfrak{h}_{\mathbb{Z}}$. Then*

$$\begin{array}{lll} \textit{Bruhat} & G = \bigsqcup_{w \in W} BwB & K = \bigsqcup_{w \in W} IwI \\ \textit{decomposition} & & \\ \\ \textit{Iwahori} & G = \bigsqcup_{w \in \widetilde{W}} IwI & G = \bigsqcup_{v \in \widetilde{W}} U^-vI \\ \textit{decomposition} & & \\ \\ \textit{Cartan} & G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+} Kt_{\lambda^\vee}K & G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^-t_{\mu^\vee}K & \textit{Iwasawa} \\ \textit{decomposition} & & & \textit{decomposition} \end{array}$$

In this paper we shall refine the Littelmann path model (in its alcove walk form, see [Ra]) by putting labels on the paths to provide a combinatorial indexing of the points in the affine flag variety. This combinatorial method of expressing the points of G/I gives detailed information about the structure of the intersections

$$U^-vI \cap IwI \quad \text{with} \quad v, w \in \widetilde{W}. \quad (1.3)$$

The corresponding intersections in G/K have arisen in many contexts. Most notably, the set of *Mirković-Vilonen cycles of shape λ^\vee and weight μ^\vee* is the set of irreducible components of the closure of $U^-t_{\mu^\vee}K \cap Kt_{\lambda^\vee}K$ in G/K ,

$$MV(\lambda^\vee)_{\mu^\vee} = \text{Irr}(\overline{U^-t_{\mu^\vee}K \cap Kt_{\lambda^\vee}K}),$$

and

$$\text{when } k = \mathbb{F}_q, \quad \text{Card}_{G/K}(U^-t_{\mu^\vee}K \cap Kt_{\lambda^\vee}K) \text{ is}$$

(up to some easily understood factors) the coefficient of the monomial symmetric function m_{μ^\vee} in the expansion of the Macdonald spherical function P_{λ^\vee} .

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2 Borchers-Kac-Moody Lie algebras

This section reviews definitions and sets notations for Borchers-Kac-Moody Lie algebras. Standard references are the book of Kac [Kac], the books of Wakimoto [Wak1][Wak2], the survey article of Macdonald [Mac3] and the handwritten notes of Macdonald [Mac2]. Specifically, [Kac, Ch. 1] is a reference for §2.1, [Kac, Ch. 3 and 5] for §2.2, and [Kac, Ch. 2] for §2.3.

2.1 Constructing a Lie algebra from a matrix

Let $A = (a_{ij})$ be an $n \times n$ matrix. Let

$$r = \text{rank}(A), \quad \ell = \text{corank}(A), \quad \text{so that} \quad r + \ell = n. \quad (2.1)$$

By rearranging rows and columns we may assume that $(a_{ij})_{1 \leq i, j \leq r}$ is nonsingular. Define a \mathbb{C} -vector space

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}, \quad \text{where} \quad \begin{array}{l} \mathfrak{h}' \text{ has basis } h_1, \dots, h_n, \quad \text{and} \\ \mathfrak{d} \text{ has basis } d_1, \dots, d_\ell. \end{array} \quad (2.2)$$

Define $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ by

$$\alpha_i(h_j) = a_{ij} \quad \text{and} \quad \alpha_i(d_j) = \delta_{i, r+j}, \quad (2.3)$$

and let

$$\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{c}, \quad \text{where} \quad \mathfrak{c} = \{h \in \mathfrak{h}' \mid \alpha_i(h) = 0 \text{ for all } 1 \leq i \leq n\}. \quad (2.4)$$

Let $c_1, \dots, c_\ell \in \mathfrak{h}'$ be a basis of \mathfrak{c} so that $h_1, \dots, h_r, c_1, \dots, c_\ell, d_1, \dots, d_\ell$ is another basis of \mathfrak{h} and define $\kappa_1, \dots, \kappa_\ell \in \mathfrak{h}^*$ by

$$\kappa_i(h_j) = 0, \quad \kappa_i(c_j) = \delta_{ij}, \quad \text{and} \quad \kappa_i(d_j) = 0. \quad (2.5)$$

Then $\alpha_1, \dots, \alpha_n, \kappa_1, \dots, \kappa_\ell$ form a basis of \mathfrak{h}^* .

Let \mathfrak{a} be the Lie algebra given by generators $\mathfrak{h}, e_1, \dots, e_n, f_1, \dots, f_n$ and relations

$$[h, h'] = 0, \quad [e_i, f_j] = \delta_{ij}h_i, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i, \quad (2.6)$$

for $h, h' \in \mathfrak{h}$ and $1 \leq i, j \leq n$. The *Borcherds-Kac-Moody Lie algebra* of A is

$$\mathfrak{g} = \frac{\mathfrak{a}}{\mathfrak{r}}, \quad \text{where} \quad \mathfrak{r} \text{ is the largest ideal of } \mathfrak{a} \text{ such that } \mathfrak{r} \cap \mathfrak{h} = 0. \quad (2.7)$$

The Lie algebra \mathfrak{a} is graded by

$$Q = \sum_{i=1}^n \mathbb{Z}\alpha_i, \quad \text{by setting} \quad \deg(e_i) = \alpha_i, \quad \deg(f_i) = -\alpha_i, \quad \deg(h) = 0, \quad (2.8)$$

for $h \in \mathfrak{h}$. Any ideal of \mathfrak{a} is Q -graded and so \mathfrak{g} is Q -graded (see [Mac2, (1.6)] or [Mac3, p. 81]),

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right), \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x\}, \quad \text{and} \quad (2.9)$$

$$R = \{\alpha \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\} \quad \text{is the set of roots of } \mathfrak{g}.$$

The *multiplicity* of a root $\alpha \in R$ is $\dim(\mathfrak{g}_\alpha)$ and the decomposition of \mathfrak{g} in (2.9) is the decomposition of \mathfrak{g} as an \mathfrak{h} -module (under the adjoint action). If

$$\begin{aligned} \mathfrak{n}^+ &\text{ is the subalgebra generated by } e_1, \dots, e_n, \text{ and} \\ \mathfrak{n}^- &\text{ is the subalgebra generated by } f_1, \dots, f_n, \end{aligned}$$

then (see [Mac3, p. 83] or [Kac, §1.3])

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{and} \quad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}, \quad (2.10)$$

where

$$R^+ = Q^+ \cap R \quad \text{with} \quad Q^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i. \quad (2.11)$$

Let \mathfrak{c} and \mathfrak{d} be as in (2.2) and (2.4). Then

$$\begin{aligned} \mathfrak{d} &\text{ acts on } \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \text{ by derivations,} \quad \mathfrak{c} = Z(\mathfrak{g}) = Z(\mathfrak{g}'), \\ \mathfrak{g} &= \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{a}/\mathfrak{r} = \mathfrak{g}' \rtimes \mathfrak{d}, \\ \mathfrak{g}' &= \mathfrak{n}^- \oplus \mathfrak{h}' \oplus \mathfrak{n}^+ = [\mathfrak{g}, \mathfrak{g}], \\ \bar{\mathfrak{g}}' &= \mathfrak{n}^- \oplus \bar{\mathfrak{h}}' \oplus \mathfrak{n}^+ = \mathfrak{g}'/\mathfrak{c}, \end{aligned} \quad (2.12)$$

and \mathfrak{g}' is the universal central extension of \mathfrak{g}' (see [Kac, Ex. 3.14]).

2.2 Cartan matrices, \mathfrak{sl}_2 subalgebras and the Weyl group

A *Cartan matrix* is an $n \times n$ matrix $A = (a_{ij})$ such that

$$a_{ij} \in \mathbb{Z}, \quad a_{ii} = 2, \quad a_{ij} \leq 0 \text{ if } i \neq j, \quad a_{ij} \neq 0 \text{ if and only if } a_{ji} \neq 0. \quad (2.13)$$

When A is a Cartan matrix the Lie algebra \mathfrak{g} contains many subalgebras isomorphic to \mathfrak{sl}_2 . For $1 \leq i \leq n$, the elements e_i and f_i act locally nilpotently on \mathfrak{g} (see [Mac3, p. 85] or [Mac2, (1.19)] or [Kac, Lemma 3.5]),

$$\text{span}\{e_i, f_i, h_i\} \cong \mathfrak{sl}_2, \quad \text{and} \quad \tilde{s}_i = \exp(\text{ad}e_i) \exp(-\text{ad}f_i) \exp(\text{ad}e_i) \quad (2.14)$$

is an automorphism of \mathfrak{g} (see [Kac, Lemma 3.8]). Thus \mathfrak{g} has lots of symmetry.

The *simple reflections* $s_i: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ and $s_i: \mathfrak{h} \rightarrow \mathfrak{h}$ are given by

$$s_i \lambda = \lambda - \lambda(h_i) \alpha_i \quad \text{and} \quad s_i h = h - \alpha_i(h) h_i, \quad \text{for } 1 \leq i \leq n, \quad (2.15)$$

$\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and

$$\tilde{s}_i \mathfrak{g}_\alpha = \mathfrak{g}_{s_i \alpha} \quad \text{and} \quad \tilde{s}_i h = s_i h, \quad \text{for } \alpha \in R, \quad h \in \mathfrak{h}.$$

The *Weyl group* W is the subgroup of $GL(\mathfrak{h}^*)$ (or $GL(\mathfrak{h})$) generated by the simple reflections. The simple reflections on \mathfrak{h} are reflections in the hyperplanes

$$\mathfrak{h}^{\alpha_i} = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0\}, \quad \text{and} \quad \mathfrak{c} = \mathfrak{h}^W = \bigcap_{i=1}^n \mathfrak{h}^{\alpha_i}.$$

The representation of W on \mathfrak{h} and \mathfrak{h}^* are dual so that

$$\lambda(wh) = (w^{-1}\lambda)(h), \quad \text{for } w \in W, \quad \lambda \in \mathfrak{h}^*, \quad h \in \mathfrak{h}.$$

The group W is presented by generators s_1, \dots, s_n and relations

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} \quad (2.16)$$

for pairs $i \neq j$ such that $a_{ij} a_{ji} < 4$, where $m_{ij} = 2, 3, 4, 6$ if $a_{ij} a_{ji} = 0, 1, 2, 3$, respectively (see [Mac2, (2.12)] or [Kac, Prop. 3.13]).

The *real roots* of \mathfrak{g} are the elements of the set

$$R_{\text{re}} = \bigcup_{i=1}^n W \alpha_i, \quad \text{and} \quad R_{\text{im}} = R \setminus R_{\text{re}} \quad (2.17)$$

is the set of *imaginary roots* of \mathfrak{g} . If $\alpha = w \alpha_i$ is a real root then there is a subalgebra isomorphic to \mathfrak{sl}_2 spanned by

$$e_\alpha = \tilde{w} e_i, \quad f_\alpha = \tilde{w} f_i, \quad \text{and} \quad h_\alpha = \tilde{w} h_i, \quad (2.18)$$

and $s_\alpha = w s_i w^{-1}$ is a reflection in W acting on \mathfrak{h} and \mathfrak{h}^* by

$$s_\alpha \lambda = \lambda - \lambda(h_\alpha) \alpha \quad \text{and} \quad s_\alpha h = h - \alpha(h) h_\alpha, \quad \text{respectively.} \quad (2.19)$$

Let $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}\text{-span}\{h_1, \dots, h_n, d_1, \dots, d_\ell\}$. The group W acts on $\mathfrak{h}_{\mathbb{R}}$ and the *dominant chamber*

$$C = \{\lambda^\vee \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha_i, \lambda^\vee \rangle \geq 0 \text{ for all } 1 \leq i \leq n\} \quad (2.20)$$

is a fundamental domain for the action of W on the *Tits cone*

$$X = \bigcup_{w \in W} wC = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha, h \rangle < 0 \text{ for a finite number of } \alpha \in R^+\}. \quad (2.21)$$

$X = \mathfrak{h}_{\mathbb{R}}$ if and only if W is finite (see [Kac, Prop. 3.12] and [Mac2, (2.14)]).

2.3 Symmetrizable matrices and invariant forms

A *symmetrizable matrix* is a matrix $A = (a_{ij})$ such that there exists a diagonal matrix

$$\mathcal{E} = \text{diag}(\epsilon_1, \dots, \epsilon_n), \quad \epsilon_i \in \mathbb{R}_{>0}, \quad \text{such that} \quad A\mathcal{E} \text{ is symmetric.} \quad (2.22)$$

If $\langle \cdot, \cdot \rangle: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is a \mathfrak{g} -invariant symmetric bilinear form then

$$\langle h_i, h \rangle = \langle [e_i, f_i], h \rangle = -\langle f_i, [e_i, h] \rangle = \langle f_i, \alpha_i(h)e_i \rangle = \alpha_i(h)\langle e_i, f_i \rangle,$$

so that

$$\langle h_i, h \rangle = \alpha_i(h)\epsilon_i, \quad \text{where} \quad \epsilon_i = \langle e_i, f_i \rangle. \quad (2.23)$$

Conversely, if A is a symmetrizable matrix then there is a nondegenerate invariant symmetric bilinear form on \mathfrak{g} determined by the formulas in (2.23) (see [Mac2, (3.12)] or [Kac, Theorem 2.2]).

If A is a Cartan matrix and $\langle \cdot, \cdot \rangle: \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C}$ is a W -invariant symmetric bilinear form then

$$\langle h_i, h \rangle = -\langle s_i h_i, h \rangle = -\langle h_i, s_i h \rangle = -\langle h_i, h - \alpha_i(h)h_i \rangle = -\langle h_i, h \rangle + \alpha_i(h)\langle h_i, h_i \rangle,$$

so that

$$\langle h_i, h \rangle = \alpha_i(h)\epsilon_i, \quad \text{where} \quad \epsilon_i = \frac{1}{2}\langle h_i, h_i \rangle. \quad (2.24)$$

In particular, $\alpha_i(h_j)\epsilon_i = \langle h_i, h_j \rangle = \langle h_j, h_i \rangle = \alpha_j(h_i)\epsilon_j$ so that A is symmetrizable. Conversely, if A is a symmetrizable Cartan matrix then there is a nondegenerate W -invariant symmetric bilinear form on \mathfrak{h} determined by the formulas in (2.24) (see [Mac2, (2.26)]).

If $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ then $[x_\alpha, y_\alpha] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$ and $\langle h, [x_\alpha, y_\alpha] \rangle = -\langle [x_\alpha, h], y_\alpha \rangle = \alpha(h)\langle x_\alpha, y_\alpha \rangle$, so that

$$[x_\alpha, y_\alpha] = \langle x_\alpha, y_\alpha \rangle h_\alpha^\vee, \quad \text{where} \quad \langle h, h_\alpha^\vee \rangle = \alpha(h) \text{ for all } h \in \mathfrak{h} \quad (2.25)$$

determines $h_\alpha^\vee \in \mathfrak{h}$. If $\alpha \in R_{\text{re}}$ and $e_\alpha, f_\alpha, h_\alpha$ are as in (2.18) then

$$h_\alpha = [e_\alpha, f_\alpha] = \langle e_\alpha, f_\alpha \rangle h_\alpha^\vee \quad \text{and} \quad \langle e_\alpha, f_\alpha \rangle = \frac{1}{2}\langle h_\alpha, h_\alpha \rangle. \quad (2.26)$$

Let

$$\alpha^\vee = \langle e_\alpha, f_\alpha \rangle \alpha = \frac{1}{2}\langle h_\alpha, h_\alpha \rangle \alpha \quad \text{so that} \quad \alpha^\vee(h) = \langle h, h_\alpha \rangle. \quad (2.27)$$

Use the vector space isomorphism

$$\begin{array}{lll} \mathfrak{h} & \xrightarrow{\sim} & \mathfrak{h}^* \\ h & \longmapsto & \langle h, \cdot \rangle \\ h_\alpha & \longmapsto & \alpha^\vee \\ h_\alpha^\vee & \longmapsto & \alpha \end{array} \quad \text{to identify} \quad Q^\vee = \sum_{i=1}^n \mathbb{Z}h_i \quad \text{and} \quad Q^* = \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee \quad (2.28)$$

and write

$$\langle \lambda^\vee, \mu \rangle = \mu(h_\lambda) \quad \text{if} \quad \lambda^\vee = \lambda_1 \alpha_1^\vee + \dots + \lambda_n \alpha_n^\vee \quad \text{and} \quad h_\lambda = \lambda_1 h_1 + \dots + \lambda_n h_n. \quad (2.29)$$

3 Steinberg-Chevalley groups

This section gives a brief treatment of the theory of Chevalley groups. The primary reference is [St] and the extensions to the Kac-Moody case are found in [Ti].

Let A be a Cartan matrix and let R_{re} be the real roots of the corresponding Borchers-Kac-Moody Lie algebra \mathfrak{g} . Let U be the enveloping algebra of \mathfrak{g} . For each $\alpha \in R_{\text{re}}$ fix a choice of e_α in (2.18) (a choice of \tilde{w}). Use the notation

$$x_\alpha(t) = \exp(te_\alpha) = 1 + e_\alpha + \frac{1}{2!}t^2e_\alpha^2 + \frac{1}{3!}t^3e_\alpha^3 + \cdots, \quad \text{in } U[[t]].$$

Then

$$x_\alpha(t)x_\alpha(u) = x_\alpha(t+u) \quad \text{in } U[[t, u]].$$

Following [Ti, 3.2], a *prenilpotent pair* is a pair of roots $\alpha, \beta \in R_{\text{re}}$ such that there exists $w, w' \in W$ with

$$w\alpha, w\beta \in R_{\text{re}}^+ \quad \text{and} \quad w'\alpha, w'\beta \in -R_{\text{re}}^+.$$

This condition guarantees that the Lie subalgebra of \mathfrak{g} generated by \mathfrak{g}_α and \mathfrak{g}_β is nilpotent. Let α, β be a prenilpotent pair and let $e_\alpha \in \mathfrak{g}_\alpha$ and $e_\beta \in \mathfrak{g}_\beta$ be as in (2.18). By [St, Lemma 15] there are unique integers $C_{\alpha\beta}^{i,j}$ such that

$$x_\alpha(t)x_\beta(u) = x_\beta(u)x_\alpha(t)x_{\alpha+\beta}(C_{\alpha,\beta}^{1,1}tu)x_{2\alpha+\beta}(C_{\alpha,\beta}^{2,1}t^2u)x_{\alpha+2\beta}(C_{\alpha,\beta}^{1,2}ut^2)\cdots.$$

Let \mathbb{F} be a commutative ring. The *Steinberg group*

$$\text{St is given by generators } x_\alpha(f) \text{ for } \alpha \in R_{\text{re}}, f \in \mathbb{F},$$

and relations

$$x_\alpha(f_1)x_\alpha(f_2) = x_\alpha(f_1 + f_2), \quad \text{for } \alpha \in R_{\text{re}}, \text{ and} \quad (3.1)$$

$$x_\alpha(f_1)x_\beta(f_2) = x_\beta(f_2)x_\alpha(f_1)x_{\alpha+\beta}(C_{\alpha,\beta}^{1,1}f_1f_2)x_{2\alpha+\beta}(C_{\alpha,\beta}^{2,1}f_1^2f_2)x_{\alpha+2\beta}(C_{\alpha,\beta}^{1,2}f_1f_2^2)\cdots \quad (3.2)$$

for prenilpotent pairs α, β . In St define

$$n_\alpha(g) = x_\alpha(g)x_{-\alpha}(-g^{-1})x_\alpha(g), \quad n_\alpha = n_\alpha(1), \quad \text{and} \quad h_{\alpha^\vee}(g) = n_\alpha(g)n_\alpha^{-1}, \quad (3.3)$$

for $\alpha \in R_{\text{re}}$ and $g \in \mathbb{F}^\times$.

Let $\mathfrak{h}_\mathbb{Z}$ be a \mathbb{Z} -lattice in \mathfrak{h} which is stable under the W -action and such that

$$\mathfrak{h}_\mathbb{Z} \supseteq Q^\vee, \quad \text{where} \quad Q^\vee = \mathbb{Z}\text{-span}\{h_1, \dots, h_n\}$$

with h_1, \dots, h_n as in (2.2). With

T given by generators $h_{\lambda^\vee}(g)$ for $\lambda^\vee \in \mathfrak{h}_\mathbb{Z}$, $g \in \mathbb{F}^\times$, and relations

$$h_{\lambda^\vee}(g_1)h_{\lambda^\vee}(g_2) = h_{\lambda^\vee}(g_1g_2) \quad \text{and} \quad h_{\lambda^\vee}(g)h_{\mu^\vee}(g) = h_{\lambda^\vee + \mu^\vee}(g), \quad (3.4)$$

the *Tits group*

$$G \text{ is the group generated by } \text{St} \text{ and } T$$

with the relations coming from the third equation in (3.3) and the additional relations

$$h_{\lambda^\vee}(g)x_\alpha(f)h_{\lambda^\vee}(g)^{-1} = x_\alpha(g^{\langle \lambda^\vee, \alpha \rangle}f) \quad \text{and} \quad n_i h_{\lambda^\vee}(g) n_i^{-1} = h_{s_i \lambda^\vee}(g). \quad (3.5)$$

For $\alpha, \beta \in R_{\text{re}}$ let $\epsilon_{\alpha\beta} = \pm 1$ be given by

$$\tilde{s}_\alpha(e_\beta) = \epsilon_{\alpha\beta} e_{s_\alpha\beta}, \quad \text{where} \quad \tilde{s}_\alpha = \exp(\text{ade}_\alpha) \exp(-\text{adf}_\alpha) \exp(\text{ade}_\alpha)$$

(see [CC, p.48] and [Ti, (3.3)]). By [St, Lemma 37] (see also [Ti, §3.7(a)])

$$n_\alpha(g)x_\beta(f)n_\alpha(g)^{-1} = x_{s_\alpha\beta}(\epsilon_{\alpha\beta}g^{-\langle\beta, \alpha^\vee\rangle}f), \quad h_{\lambda^\vee}(g)x_\beta(f)h_{\lambda^\vee}(g)^{-1} = x_\beta(g^{\langle\beta, \lambda^\vee\rangle}f), \quad (3.6)$$

$$\text{and} \quad n_\alpha(g)h_{\lambda^\vee}(g')n_\alpha(g)^{-1} = h_{s_\alpha\lambda^\vee}(g'). \quad (3.7)$$

Thus G has a symmetry under the subgroup

$$N \text{ generated by } T \text{ and the } n_\alpha(g) \text{ for } \alpha \in R_{\text{re}}, g \in \mathbb{F}^\times. \quad (3.8)$$

If \mathbb{F} is big enough then N is the normalizer of T in G [St, Ex. (b) p. 36] and, by [St, Lemma 27], the homomorphism

$$\begin{array}{ccc} N & \longrightarrow & W \\ n_\alpha(g) & \longmapsto & s_\alpha \end{array} \quad \text{is surjective with kernel } T. \quad (3.9)$$

Remark 3.1. [Ti, §3.7(b)] If $\mathfrak{h}_\mathbb{Z} = Q^\vee$ and the first relation of (3.5) holds in St then there is a surjective homomorphism $\psi: \text{St} \rightarrow G$. By [St, Lemma 22], the elements

$$n_\alpha h_{\lambda^\vee}(g) n_\alpha^{-1} h_{s_\alpha \lambda^\vee}(g)^{-1} \quad \text{and} \quad n_\alpha(g) n_\alpha^{-1} h_{\alpha^\vee}(g)^{-1}$$

automatically commute with each $x_\beta(f)$ so that $\ker(\psi) \subseteq Z(\text{St})$. In many cases St is the universal central extension of G (see [Ti, 3.7(c)] and [St, Theorems 10,11,12]).

Remark 3.2. The algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ in (2.12) is generated by e_α , $\alpha \in R_{\text{re}}$. A \mathfrak{g}' -module V is *integrable* if e_α , $\alpha \in R_{\text{re}}$, act locally nilpotently so that

$$x_\alpha(c) = \exp(ce_\alpha), \quad \text{for } \alpha \in R_{\text{re}}, c \in \mathbb{C}, \quad (3.10)$$

are well defined operators on V . The *Chevalley group* G_V is the subgroup of $GL(V)$ generated by the operators in (3.10). To do this integrally use a Kostant \mathbb{Z} -form and choose a lattice in the module V (see [Ti, §4.3-4] and [St, Ch. 1]). The *Kac-Moody group* is the group G_{KM} generated by symbols

$$x_\alpha(c), \quad \alpha \in R_{\text{re}}, c \in \mathbb{C}, \quad \text{with relations} \quad x_\alpha(c_1)x_\alpha(c_2) = x_\alpha(c_1 + c_2)$$

and the additional relations coming from forcing an element to be 1 if it acts by 1 on *every* integrable \mathfrak{g}' module. This is essentially the Chevalley group G_V for the case when V is the adjoint representation and so $G_{KM} \subseteq \text{Aut}(\mathfrak{g}')$. There are surjective homomorphisms

$$\text{St}(\mathbb{C}) \rightarrow G_{KM} \rightarrow G_V.$$

See [Kac, Exercises 3.16-19] and [Ti, Proposition 1].

Remark 3.3. [St, Lemma 28] In the setting of Remark 3.2 let T_V be the subgroup of G_V generated by $h_{\alpha^\vee}(g)$ for $\alpha \in R_{\text{re}}, g \in \mathbb{F}^\times$. Then

$$h_{\alpha_1^\vee}(g_1) \cdots h_{\alpha_n^\vee}(g_n) = 1 \quad \text{if and only if} \quad g_1^{\langle\mu, \alpha_1^\vee\rangle} \cdots g_n^{\langle\mu, \alpha_n^\vee\rangle} = 1 \quad \text{for all weights } \mu \text{ of } V,$$

$$Z(G_V) = \{h_{\alpha_1^\vee}(g_1) \cdots h_{\alpha_n^\vee}(g_n) \mid g_1^{\langle\beta, \alpha_1^\vee\rangle} \cdots g_n^{\langle\beta, \alpha_n^\vee\rangle} = 1 \quad \text{for all } \beta \in R\},$$

and if \mathbb{F} is big enough

$$T_V = \{h_{\omega_1^\vee}(g_1) \cdots h_{\omega_n^\vee}(g_n) \mid g_1, \dots, g_n \in \mathbb{F}^\times\},$$

where $\omega_1^\vee, \dots, \omega_n^\vee$ is a \mathbb{Z} -basis of the \mathbb{Z} -span of the weights of V [St, Lemma 35].

4 Labeling points of the flag variety G/B

In this section we follow [St, Ch.8] to show that the points of the flag variety are naturally indexed by labeled walks. This is the first step in making a precise connection between the points in the flag variety and the alcove walk theory in [Ra].

Let G be a Tits group as in (3.5) over the field $\mathbb{F} = \mathbb{C}$. The *root subgroups*

$$\mathcal{X}_\alpha = \{x_\alpha(c) \mid c \in \mathbb{C}\}, \quad \text{for } \alpha \in R_{\text{re}}, \quad \text{satisfy} \quad w\mathcal{X}_\beta w^{-1} = \mathcal{X}_{w\beta}, \quad (4.1)$$

for $w \in W$ and $\beta \in R_{\text{re}}$, since $h_{\alpha^\vee}(c)\mathcal{X}_\beta h_{\alpha^\vee}(c)^{-1} = \mathcal{X}_\beta$ and $n_\alpha \mathcal{X}_\beta n_\alpha^{-1} = \mathcal{X}_{s_\alpha \beta}$. As a group \mathcal{X}_α is isomorphic to \mathbb{C} (under addition).

The *flag variety* is G/B , where the subgroup

$$B \text{ is generated by } T \text{ and } x_\alpha(f) \text{ for } \alpha \in R_{\text{re}}^+, f \in \mathbb{C}. \quad (4.2)$$

Let $w \in W$. The *inversion set* of w is

$$R(w) = \{\alpha \in R_{\text{re}}^+ \mid w^{-1}\alpha \notin R_{\text{re}}^+\} \quad \text{and} \quad \ell(w) = \text{Card}(R(w)) \quad (4.3)$$

is the *length* of w . View a reduced expression $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ in the generators in (2.16) as a *walk* in W starting at 1 and ending at w ,

$$1 \longrightarrow s_{i_1} \longrightarrow s_{i_1} s_{i_2} \longrightarrow \cdots \longrightarrow s_{i_1} \cdots s_{i_\ell} = w. \quad (4.4)$$

Letting $x_i(c) = x_{\alpha_i}(c)$ and $n_i = n_{\alpha_i}(1)$, the following theorem shows that

$$BwB = \{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B \mid c_1, \dots, c_\ell \in \mathbb{C}\} \quad (4.5)$$

so that the G/B -points of BwB are in bijection with labelings of the edges of the walk by complex numbers c_1, \dots, c_ℓ . The elements of $R(w)$ are

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}\alpha_{i_2}, \quad \dots, \quad \beta_\ell = s_{i_1} \cdots s_{i_{\ell-1}}\alpha_{i_\ell}, \quad (4.6)$$

and the first relation in (3.6) gives

$$x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} = x_{\beta_1}(\pm c_1) \cdots x_{\beta_\ell}(\pm c_\ell)n_w, \quad (4.7)$$

where $n_w = n_{i_1}^{-1} \cdots n_{i_\ell}^{-1}$.

Theorem 4.1. [St, Thm. 15 and Lemma 43] *Let $w \in W$ and let n_w be a representative of w in N . If*

$$R(w) = \{\beta_1, \dots, \beta_\ell\} \quad \text{then} \quad \{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell)n_w \mid c_1, \dots, c_\ell \in \mathbb{C}\}$$

is a set of representatives of the B -cosets in BwB .

Proof. The conceptual reason for this is that

$$\begin{aligned} BwB &= \left(\prod_{\alpha \in R_{\text{re}}^+} \mathcal{X}_\alpha \right) n_w B = n_w \left(\prod_{w^{-1}\alpha \notin R_{\text{re}}^+} \mathcal{X}_{w^{-1}\alpha} \right) \left(\prod_{w^{-1}\alpha \in R_{\text{re}}^+} \mathcal{X}_{w^{-1}\alpha} \right) B \\ &= n_w \left(\prod_{w^{-1}\alpha \notin R_{\text{re}}^+} \mathcal{X}_{w^{-1}\alpha} \right) B = \left(\prod_{\alpha \in R(w)} \mathcal{X}_\alpha \right) n_w B \\ &= \{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell)n_w B \mid c_1, \dots, c_\ell \in \mathbb{F}\}. \end{aligned}$$

Since R_{re}^+ may be infinite there is a subtlety in the decomposition and ordering of the product of \mathcal{X}_α in the second ‘‘equality’’ and it is necessary to proceed more carefully. Choose a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ and let $\beta_1, \dots, \beta_\ell$ be the ordering of $R(w)$ from (4.6).

Step 1: Since $R(w) \subseteq R_{\text{re}}^+$ there is an inclusion

$$\{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) n_w B \mid c_1, \dots, c_\ell \in \mathbb{C}\} \subseteq BwB.$$

To prove equality proceed by induction on ℓ .

Base case: Suppose that $w = s_j$. Let $\alpha \in R_{\text{re}}^+$ and $c, d \in \mathbb{C}$. If $c = 0$ or α, α_j is a prenilpotent pair then, by relation (3.2),

$$x_\alpha(d)x_{\alpha_j}(c)n_j^{-1}B = x_{\alpha_j}(c')n_j^{-1}B, \quad \text{for some } c' \in \mathbb{C}. \quad (4.8)$$

If α, α_j is not a prenilpotent pair and $c \neq 0$ then $\alpha, -\alpha_j$ is a prenilpotent pair and, by (3.2),

$$x_\alpha(d)x_{\alpha_j}(c)n_j^{-1}B = x_\alpha(d)x_{-\alpha_j}(c^{-1})B = x_{-\alpha_j}(c^{-1})B = x_{\alpha_j}(c)n_j^{-1}B.$$

Thus $\{x_{\alpha_j}(c)n_j^{-1}B \mid c \in \mathbb{C}\}$ is B -invariant and so $Bs_jB = \{x_{\alpha_j}(c)n_j^{-1}B \mid c \in \mathbb{C}\}$.

Induction step: If $w = s_{i_1} \cdots s_{i_\ell}$ is reduced and if $\ell(ws_j) > \ell(w)$ then, by induction,

$$Bws_jB \subseteq BwB \cdot Bs_jB = \{x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) x_{w\alpha_j}(c) n_w n_j^{-1} B \mid c_1, \dots, c_\ell, c \in \mathbb{F}\},$$

so that $Bws_jB = \{x_{\beta_1}(c_1) \cdots x_{\beta_{\ell+1}}(c_{\ell+1}) n_{ws_j} B \mid c_1, \dots, c_{\ell+1} \in \mathbb{C}\}$ with $\beta_{\ell+1} = w\alpha_j$.

Step 2: Prove that $BwB = BvB$ if and only if $w = v$ by induction on $\ell(w)$.

Base case: Suppose that $\ell(w) = 0$. Then $BwB = BvB$ implies that $v \in B$ so that there is a representative n_v of v such that $n_v \in B \cap N$. Then $vR_{\text{re}}^+ \subseteq R_{\text{re}}^+$ since $n_v \mathcal{X}_\alpha n_v^{-1} = \mathcal{X}_{v\alpha} \in B$ for $\alpha \in R_{\text{re}}^+$. So $\ell(v) = 0$. Thus, by (2.16), $v = 1$.

Induction step: Assume $BwB = BvB$ and s_j is such that $\ell(ws_j) < \ell(w)$. Since $BvB \cdot Bs_jB \subseteq BvB \cup Bvs_jB$ (see [St, Lemma 25]),

$$Bws_jB \subseteq BwB \cdot Bs_jB = BvB \cdot Bs_jB \subseteq BvB \cup Bvs_jB = BwB \cup Bvs_jB.$$

Thus, by induction, $ws_j = w$ or $ws_j = vs_j$. Since $ws_j \neq w$, it follows that $w = v$.

Step 3: Let us show that if $x_{\alpha_{i_1}}(c_1)n_{i_1}^{-1} \cdots x_{\alpha_{i_\ell}}(c_\ell)n_{i_\ell}^{-1}B = x_{\alpha_{i_1}}(c'_1)n_{i_1}^{-1} \cdots x_{\alpha_{i_\ell}}(c'_\ell)n_{i_\ell}^{-1}B$, then $c_i = c'_i$ for $i = 1, 2, \dots, \ell$. The left hand side of

$$x_{\alpha_2}(c_2)n_{i_2}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}B = n_{i_1}x_{i_1}(c'_1 - c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c'_\ell)n_{i_\ell}^{-1}B$$

is in $Bs_{i_2} \cdots s_{i_\ell}B$. If $c'_1 \neq c_1$ then $n_{i_1}^{-1}x_{i_1}(c'_1 - c_1)n_{i_1} \in Bs_{i_1}B$ and the right hand side is contained in

$$n_{i_1}^{-1}x_{i_1}(c'_1 - c_1)n_{i_1}Bs_{i_2} \cdots s_{i_\ell}B \subseteq Bs_{i_1}B \cdot Bs_{i_2} \cdots s_{i_\ell}B = Bs_{i_1} \cdots s_{i_\ell}B.$$

By Step 2 this is impossible and so $c'_1 = c_1$. Then, by induction, $c'_i = c_i$ for $i = 1, 2, \dots, \ell$.

Step 4: From the definition of $R(w)$ it follows that if $\alpha, \beta \in R(w)$ and $\alpha + \beta \in R_{\text{re}}$ then $\alpha + \beta \in R(w)$ and if $\alpha, \beta \in R(w)$ then α, β form a prenilpotent pair. Thus, by [St, Lemma 17], any total order on the set $R(w)$ can be taken in the statement of the theorem. \square

Remark 4.2. Suppose that $\lambda \in \mathfrak{h}^*$ is dominant integral and $M(\lambda)$ is an (integrable) highest weight representation of G generated by a highest weight vector v_λ^+ . Then the set $BwBv_\lambda^+$ contains the vector wv_λ^+ and is contained in the sum $\bigoplus_{\nu \geq w\lambda} M(\lambda)_\nu$ of the weight spaces with weights $\geq w\lambda$. This is another way to show that if $w \neq v$ then $BwB \neq BvB$ and accomplish Step 2 in the proof of Theorem 4.1.

5 Loop Lie algebras and their extensions

This section gives a presentation of the theory of loop Lie algebras. The main lines of the theory are exactly as in the classical case (see, for example, [Mac2, §4] and [Kac, ch. 7]) but, following recent trends (see [Ga], [GK], [GR] and [Rou]) we treat the more general setting of the loop Lie algebra of a Kac-Moody Lie algebra.

Let \mathfrak{g}_0 be a symmetrizable Kac-Moody Lie algebra with bracket $[\cdot, \cdot]_0: \mathfrak{g}_0 \otimes \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ and invariant form $\langle \cdot, \cdot \rangle_0: \mathfrak{g}_0 \times \mathfrak{g}_0 \rightarrow \mathbb{C}$. The *loop Lie algebra* is

$$\mathfrak{g}_0[t, t^{-1}] = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}_0 \quad \text{with bracket} \quad [t^m x, t^n y]_0 = t^{m+n} [x, y]_0,$$

for $x, y \in \mathfrak{g}_0$. Let

$$\mathfrak{g} = \mathfrak{g}_0[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \mathfrak{g}' = \mathfrak{g}_0[t, t^{-1}] \oplus \mathbb{C}c, \quad \bar{\mathfrak{g}}' = \mathfrak{g}_0[t, t^{-1}] = \frac{\mathfrak{g}'}{\mathbb{C}c}$$

where the bracket on \mathfrak{g} is given by

$$[t^m x, t^n y] = t^{m+n} [x, y]_0 + \delta_{m+n, 0} m \langle x, y \rangle_0 c, \quad c \in Z(\mathfrak{g}), \quad [d, t^m x] = m t^m x. \quad (5.1)$$

By [Kac, Ex. 7.8], \mathfrak{g}' is the universal central extension of $\bar{\mathfrak{g}}'$. An invariant symmetric form on \mathfrak{g} is given by

$$\langle c, d \rangle = 1, \quad \langle c, t^m y \rangle = \langle d, t^m y \rangle = 0, \quad \langle c, c \rangle = \langle d, d \rangle = 0, \quad (5.2)$$

and

$$\langle t^m x, t^n y \rangle = \begin{cases} \langle x, y \rangle_0, & \text{if } m+n=0, \\ 0, & \text{otherwise,} \end{cases} \quad (5.3)$$

for $x, y \in \mathfrak{g}_0$, $m, n \in \mathbb{Z}$.

Fix a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 and let

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \mathfrak{h}' = \mathfrak{h}_0 \oplus \mathbb{C}c, \quad \bar{\mathfrak{h}}' = \mathfrak{h}_0. \quad (5.4)$$

As in (2.2), let $h_1, \dots, h_n, d_1, \dots, d_\ell$ be a basis of \mathfrak{h}_0 and let

$$\{h_1, \dots, h_n, d_1, \dots, d_\ell, c, d\} \text{ be a basis of } \mathfrak{h} \text{ and} \quad (5.5)$$

$$\{\omega_1, \dots, \omega_n, \delta_1, \dots, \delta_\ell, \Lambda_0, \delta\} \text{ the dual basis in } \mathfrak{h}^*$$

so that

$$\delta(\mathfrak{h}_0) = 0, \quad \delta(c) = 0, \quad \delta(d) = 1, \quad \text{and} \quad \Lambda_0(\mathfrak{h}_0) = 0, \quad \Lambda_0(c) = 1, \quad \Lambda_0(d) = 0. \quad (5.6)$$

Let R be as in (2.9). As an \mathfrak{h} -module

$$\mathfrak{g} = \left(\bigoplus_{\substack{\alpha \in R \\ k \in \mathbb{Z}}} \mathfrak{g}_{\alpha+k\delta} \right) \oplus \left(\bigoplus_{k \in \mathbb{Z} \neq 0} \mathfrak{g}_{k\delta} \right) \oplus \mathfrak{h}, \quad \text{where } \mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad (5.7)$$

$$\mathfrak{g}_{\alpha+k\delta} = t^k \mathfrak{g}_\alpha, \quad \mathfrak{g}_{k\delta} = t^k \mathfrak{h}_0, \quad \text{and} \quad \tilde{R} = (R + \mathbb{Z}\delta) \cup \mathbb{Z}_{\neq 0}\delta \quad (5.8)$$

is the set of *roots* of \mathfrak{g} .

Let $\alpha \in R_{\text{re}}$ with $\alpha = w\alpha_i$ and fix a choice of e_α, f_α and h_α in (2.18) (choose \tilde{w}). Then

$$e_{-\alpha+k\delta} = t^k f_\alpha, \quad f_{-\alpha+k\delta} = t^{-k} e_\alpha, \quad h_{-\alpha+k\delta} = -h_\alpha + k \langle e_\alpha, f_\alpha \rangle_0 c, \quad (5.9)$$

span a subalgebra isomorphic to \mathfrak{sl}_2 . If $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ is the decomposition in (2.10) and

\mathfrak{n}^+ is the subalgebra generated by \mathfrak{n}_0^+ and $e_{-\alpha+k\delta}$ for $\alpha \in R_{\text{re}}$, $k \in \mathbb{Z}_{>0}$, and
 \mathfrak{n}^- is the subalgebra generated by \mathfrak{n}_0^- and $f_{-\alpha+k\delta}$ for $\alpha \in R_{\text{re}}$, $k \in \mathbb{Z}_{>0}$,

then

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{with} \quad \mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \left(\bigoplus_{\substack{\alpha \in R \cup \{0\} \\ k \in \mathbb{Z}_{>0}}} \mathfrak{g}_{\alpha+k\delta} \right) \quad \text{and} \quad \mathfrak{n}^- = \mathfrak{n}_0^- \oplus \left(\bigoplus_{\substack{\alpha \in R \cup \{0\} \\ k \in \mathbb{Z}_{<0}}} \mathfrak{g}_{\alpha+k\delta} \right).$$

The elements $e_{-\alpha+k\delta}$ and $f_{-\alpha+k\delta}$ in (5.9) act locally nilpotently on \mathfrak{g} because f_α and e_α act locally nilpotently on \mathfrak{g}_0 . Thus

$$\tilde{s}_{-\alpha+k\delta} = \exp(\text{ad } t^k f_\alpha) \exp(-\text{ad } t^{-k} e_\alpha) \exp(\text{ad } t^k f_\alpha) \quad (5.10)$$

is a well defined automorphism of \mathfrak{g} and

$$\tilde{s}_{-\alpha+k\delta} \mathfrak{g} \beta = \mathfrak{g}_{s_{-\alpha+k\delta} \beta} \quad \text{and} \quad \tilde{s}_{-\alpha+k\delta} h = s_{-\alpha+k\delta} h, \quad (5.11)$$

for $h \in \mathfrak{h}$ and $\beta \in \tilde{R}$, where $s_{-\alpha+k\delta}: \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ and $s_{-\alpha+k\delta}: \mathfrak{h} \rightarrow \mathfrak{h}$ are given by

$$s_{-\alpha+k\delta} \lambda = \lambda - \lambda(h_{-\alpha+k\delta})(-\alpha + k\delta) \quad \text{and} \quad s_{-\alpha+k\delta} h = h - (-\alpha + k\delta)(h)h_{-\alpha+k\delta}, \quad (5.12)$$

for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. The *Weyl group* of \mathfrak{g} is the subgroup of $GL(\mathfrak{h}^*)$ (or $GL(\mathfrak{h})$) generated by the reflections $s_{-\alpha+k\delta}$,

$$W_{\text{aff}} = \langle s_{-\alpha+k\delta} \mid \alpha \in R_{\text{re}}, k \in \mathbb{Z} \rangle. \quad (5.13)$$

Noting that $\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ and $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$, use (5.12) to compute

$$\begin{aligned} s_{-\alpha+k\delta}(\bar{\lambda}) &= \bar{\lambda} + \bar{\lambda}(h_\alpha)(-\alpha + k\delta), & s_{-\alpha+k\delta}(\bar{h}) &= \bar{h} + \alpha(\bar{h})(-h_\alpha + k\langle e_\alpha, f_\alpha \rangle_0 c), \\ s_{-\alpha+k\delta}(\ell\Lambda_0) &= \ell\Lambda_0 - k\ell\langle e_\alpha, f_\alpha \rangle_0(-\alpha + k\delta), & s_{-\alpha+k\delta}(mc) &= mc, \\ s_{-\alpha+k\delta}(m\delta) &= m\delta, & s_{-\alpha+k\delta}(\ell d) &= \ell d - k\ell(-h_\alpha + k\langle e_\alpha, f_\alpha \rangle_0 c). \end{aligned}$$

for $\bar{\lambda} \in \mathfrak{h}_0^*$, $\bar{h} \in \mathfrak{h}_0$, $m, \ell \in \mathbb{C}$. For $\alpha \in R_{\text{re}}$ and $k \in \mathbb{Z}$

$$\text{define } t_{k\alpha^\vee} \in W_{\text{aff}} \text{ by} \quad s_{-\alpha+k\delta} = t_{k\alpha^\vee} s_{-\alpha}, \quad (5.14)$$

and use (2.26) and (2.27) to compute

$$\begin{aligned} t_{k\alpha^\vee}(\bar{\lambda}) &= \bar{\lambda} - \bar{\lambda}(kh_\alpha)\delta, & t_{k\alpha^\vee}(\bar{h}) &= \bar{h} - k\alpha^\vee(\bar{h})c, \\ t_{k\alpha^\vee}(\ell\Lambda_0) &= \ell\Lambda_0 + \ell k\alpha^\vee - \ell \frac{1}{2} \langle kh_\alpha, kh_\alpha \rangle_0 \delta, & t_{k\alpha^\vee}(mc) &= mc, \\ t_{k\alpha^\vee}(m\delta) &= m\delta, & t_{k\alpha^\vee}(\ell d) &= \ell d + \ell kh_\alpha - \ell \frac{1}{2} \langle kh_\alpha, kh_\alpha \rangle_0 c. \end{aligned}$$

Then $t_{k\alpha^\vee} t_{j\beta^\vee}(\bar{\lambda}) = t_{kh_\alpha}(\bar{\lambda} - \bar{\lambda}(jh_\beta)\delta) = \bar{\lambda} - \bar{\lambda}(kh_\alpha + jh_\beta)\delta$, and

$$\begin{aligned} t_{k\alpha^\vee} t_{j\beta^\vee}(\ell\Lambda_0) &= t_{k\alpha^\vee}(\ell\Lambda_0 + \ell j\beta^\vee - \ell \frac{1}{2} \langle jh_\beta, jh_\beta \rangle_0 \delta) \\ &= \ell\Lambda_0 + \ell k\alpha^\vee - \ell \frac{1}{2} \langle kh_\alpha, kh_\alpha \rangle_0 \delta + \ell j\beta^\vee - \ell j\beta^\vee(kh_\alpha)\delta - \ell \frac{1}{2} \langle jh_\beta, jh_\beta \rangle_0 \delta \\ &= \ell\Lambda_0 + \ell(k\alpha^\vee + j\beta^\vee) - \ell \frac{1}{2} \langle kh_\alpha + jh_\beta, kh_\alpha + jh_\beta \rangle_0 \delta. \end{aligned}$$

This computation shows that $t_{k\alpha^\vee} t_{j\beta^\vee} = t_{j\alpha^\vee + k\beta^\vee}$. Thus, if W_0 is the Weyl group of \mathfrak{g}_0 and $Q^* = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ then

$$W_{\text{aff}} = \{t_{\lambda^\vee} w \mid \lambda^\vee \in Q^*, w \in W_0\} \quad \text{with} \quad t_{\lambda^\vee} t_{\mu^\vee} = t_{\lambda^\vee + \mu^\vee} \quad \text{and} \quad w t_{\lambda^\vee} = t_{w\lambda^\vee} w, \quad (5.15)$$

for $w \in W_0$, $\lambda^\vee, \mu^\vee \in Q^*$.

Since $\mathbb{C}\delta$ is W_{aff} -invariant, the group W_{aff} acts on $\mathfrak{h}^*/\mathbb{C}\delta$ and W_{aff} acts on the set

$$\begin{array}{ccc} (\mathfrak{h}_0^* + \Lambda_0 + \mathbb{C}\delta)/\mathbb{C}\delta & \xrightarrow{\sim} & \mathfrak{h}_0^* \\ \bar{\lambda} + \Lambda_0 + \mathbb{C}\delta & \mapsto & \bar{\lambda} \end{array} \quad (5.16)$$

and the W_{aff} -action on the right hand side is given by

$$s_\alpha(\bar{\lambda}) = \bar{\lambda} - \bar{\lambda}(h_\alpha)\alpha \quad \text{and} \quad t_{k\alpha^\vee}(\bar{\lambda}) = \bar{\lambda} + k\alpha^\vee, \quad \text{for } \bar{\lambda} \in \mathfrak{h}_0. \quad (5.17)$$

Here \mathfrak{h}_0^* is a set with a W_{aff} -action, the action of W_{aff} is *not linear*.

6 Loop groups and the affine flag variety G/I

This section gives a short treatment of loop groups following [St, Ch. 8] and [Mac1, §2.5 and 2.6]. This theory is currently a subject of intense research as evidenced by the work in [Ga], [GK], [Rem], [Rou], [GR].

Let \mathfrak{g}_0 be a symmetrizable Kac-Moody Lie algebra and let $\mathfrak{h}_{\mathbb{Z}}$ be a \mathbb{Z} -lattice in \mathfrak{h}_0 that contains $Q^\vee = \mathbb{Z}\text{-span}\{h_1, \dots, h_n\}$.

$$\text{The loop group is the Tits group } G = G_0(\mathbb{C}((t))) \quad (6.1)$$

over the field $\mathbb{F} = \mathbb{C}((t))$. Let $K = G_0(\mathbb{C}[[t]])$ and $G_0(\mathbb{C})$ be the Tits groups of \mathfrak{g}_0 and $\mathfrak{h}_{\mathbb{Z}}$ over the rings $\mathbb{C}[[t]]$ and \mathbb{C} , respectively, and let $B(\mathbb{C})$ be the standard *Borel subgroup* of $G_0(\mathbb{C})$ as defined in (4.2). Let

$$U^- \text{ be the subgroup of } G \text{ generated by } x_{-\alpha}(f) \text{ for } \alpha \in R_{\text{re}}^+ \text{ and } f \in \mathbb{C}((t)), \quad (6.2)$$

and define the standard *Iwahori subgroup* I of G by

$$\begin{array}{rcc} G & = & G_0(\mathbb{C}((t))) \\ \cup & & \cup \\ K & = & G_0(\mathbb{C}[[t]]) \xrightarrow{\text{ev}_{t=0}} G_0(\mathbb{C}) \\ \cup & & \cup \quad \cup \\ I & = & \text{ev}_{t=0}^{-1}(B(\mathbb{C})) \xrightarrow{\text{ev}_{t=0}} B(\mathbb{C}). \end{array} \quad (6.3)$$

The *affine flag variety* is G/I .

For $\alpha + j\delta \in R_{\text{re}} + \mathbb{Z}\delta$ and $c \in \mathbb{C}$, define

$$x_{\alpha+j\delta}(c) = x_\alpha(ct^j) \quad \text{and} \quad t_{\lambda^\vee} = h_{\lambda^\vee}(t^{-1}), \quad (6.4)$$

and, for $c \in \mathbb{C}^\times$, define

$$n_{\alpha+j\delta}(c) = x_{\alpha+j\delta}(c)x_{-\alpha-j\delta}(-c^{-1})x_{\alpha+j\delta}(c), \quad (6.5)$$

$$n_{\alpha+j\delta} = n_{\alpha+j\delta}(1), \quad \text{and} \quad h_{(\alpha+j\delta)^\vee}(c) = n_{\alpha+j\delta}(c)n_{\alpha+j\delta}^{-1} \quad (6.6)$$

analogous to (3.3).

The group

$$\widetilde{W} = \{t_{\lambda^\vee}w \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, w \in W_0\} \quad \text{with} \quad t_{\lambda^\vee}t_{\mu^\vee} = t_{\lambda^\vee+\mu^\vee} \quad \text{and} \quad wt_{\lambda^\vee} = t_{w\lambda^\vee}w, \quad (6.7)$$

acts on $\mathfrak{h}_0^* \oplus \mathbb{C}\delta$ by

$$v(\mu + k\delta) = v\mu + k\delta \quad \text{and} \quad t_{\lambda^\vee}(\mu + k\delta) = \mu + (k - \langle \lambda^\vee, \mu \rangle)\delta \quad (6.8)$$

for $v \in W_0$, $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$, $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, and $k \in \mathbb{Z}$. Then $n_{\alpha+j\delta}(c) = t_{-j\alpha^\vee}n_\alpha(c) = n_\alpha(ct^j)$,

$$n_\alpha x_{\beta+k\delta}(c)n_\alpha^{-1} = n_\alpha x_{\beta}(ct^k)n_\alpha^{-1} = x_{s_\alpha\beta}(\epsilon_{\alpha,\beta}ct^k) = x_{s_\alpha(\beta+k\delta)}(\epsilon_{\alpha,\beta}c)$$

for $\alpha \in R_{\text{re}}$, and, for $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$,

$$t_{\lambda^\vee}x_{\beta+k\delta}(c)t_{\lambda^\vee}^{-1} = x_{\beta+k\delta}(t^{-\langle \lambda^\vee, \beta \rangle}c) = x_{t_{\lambda^\vee}(\beta+k\delta)}(c).$$

Thus the *root subgroups*

$$\mathcal{X}_{\alpha+j\delta} = \{x_{\alpha+j\delta}(c) \mid c \in \mathbb{C}\} \quad \text{satisfy} \quad w\mathcal{X}_{\alpha+j\delta}w^{-1} = \mathcal{X}_{w(\alpha+j\delta)} \quad (6.9)$$

for $w \in \widetilde{W}$ and $\alpha + j\delta \in R_{\text{re}} + \mathbb{Z}\delta$. These relations are a reflection of the symmetry of the group G under the group defined in (3.8):

$$\widetilde{N} = N(\mathbb{C}((t))) \quad \text{generated by } n_\alpha(g), h_{\lambda^\vee}(g), \text{ for } g \in \mathbb{C}((t))^\times, \quad (6.10)$$

$\alpha \in R_{\text{re}}$, and $\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}$. The homomorphism $\widetilde{N} \rightarrow W_0$ from (3.9) lifts to a surjective homomorphism (see [Mac1, p. 26 and p. 28])

$$\begin{array}{ccc} \widetilde{N} & \longrightarrow & \widetilde{W} \\ n_{\alpha+j\delta} & \longmapsto & t_{-j\alpha^\vee}s_\alpha \\ t_{\lambda^\vee} & \longmapsto & t_{\lambda^\vee}^\vee \end{array} \quad \text{with kernel } H \text{ generated by } h_\lambda(d), d \in \mathbb{C}[[t]]^\times.$$

Define

$$\tilde{R}_{\text{re}}^I = (R_{\text{re}}^+ + \mathbb{Z}_{\geq 0}\delta) \sqcup (-R_{\text{re}}^+ + \mathbb{Z}_{> 0}\delta) \quad \text{and} \quad \tilde{R}_{\text{re}}^U = -R_{\text{re}}^+ + \mathbb{Z}\delta \quad (6.11)$$

so that

$$\begin{array}{l} \mathcal{X}_{\alpha+j\delta} \subseteq I \quad \text{if and only if} \quad \alpha + j\delta \in \tilde{R}_{\text{re}}^I \quad \text{and} \\ \mathcal{X}_{\alpha+j\delta} \subseteq U^- \quad \text{if and only if} \quad \alpha + j\delta \in \tilde{R}_{\text{re}}^U. \end{array} \quad (6.12)$$

Note that $\tilde{R}_{\text{re}}^I \sqcup (-\tilde{R}_{\text{re}}^I) = \tilde{R}_{\text{re}}^U \sqcup (-\tilde{R}_{\text{re}}^U) = R_{\text{re}} + \mathbb{Z}\delta$.

7 The folding algorithm and the intersections $U^-vI \cap IwI$

In this section we prove our main theorem, which gives a precise connection between the alcove walks in [Ra] and the points in the affine flag variety. The algorithm here is essentially that which is found in [BD] and, with our setup from the earlier sections, it is the ‘obvious one’. The same method has, of course, been used in other contexts, see, for example, [C].

A special situation in the loop group theory is when \mathfrak{g}_0 is finite dimensional. In this case, the extended loop Lie algebra \mathfrak{g} defined in (5.1) is also a Kac-Moody Lie algebra. If G_0 is the Tits group of \mathfrak{g}_0 and $G = G_0(\mathbb{C}((t)))$ is the corresponding loop group then the subgroup I defined in (6.3) differs from the Borel subgroup of the Kac-Moody group G_{KM} for \mathfrak{g} only by elements of T , and the affine flag variety of G coincides with the flag variety of G_{KM} . Thus, in this case, Theorem 4.1 provides a labeling of the points of the affine flag variety.

Suppose that \mathfrak{g}_0 is a finite dimensional complex semisimple Lie algebra presented as a Kac-Moody Lie algebra with generators $e_1, \dots, e_n, f_1, \dots, f_n, h_1, \dots, h_n$ and Cartan matrix $A =$

$(\alpha_i(h_j))_{1 \leq i, j \leq n}$. Let φ be the highest root of R (the highest weight of the adjoint representation), fix

$$e_\varphi \in \mathfrak{g}_\varphi, \quad f_\varphi \in \mathfrak{g}_{-\varphi} \quad \text{such that} \quad \langle e_\varphi, f_\varphi \rangle_0 = 1,$$

and let

$$e_0 = e_{-\varphi+\delta} = t f_\varphi, \quad f_0 = f_{-\varphi+\delta} = t^{-1} e_\varphi, \quad h_0 = [e_0, f_0] = [t x_{-\varphi}, t^{-1} x_\varphi] = -h_\varphi + c,$$

as in (5.9). The magical fact is that, in this case, $\mathfrak{g} = \mathfrak{g}_0[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Kac-Moody Lie algebra with generators $e_0, \dots, e_n, f_0, \dots, f_n, h_0, \dots, h_n, d$ and Cartan matrix

$$A^{(1)} = (\alpha_i(h_j))_{0 \leq i, j \leq n}, \quad \text{where} \quad \alpha_0 = -\varphi + \delta \quad \text{and} \quad h_0 = -h_\varphi + c, \quad (7.1)$$

where δ is as in (5.6) (see [Kac, Thm. 7.4]).

The *alcoves* are the open connected components of

$$\mathfrak{h}_\mathbb{R} \setminus \bigcup_{-\alpha+j\delta \in \tilde{R}_{\text{re}}^+} H_{-\alpha+j\delta}, \quad \text{where} \quad H_{-\alpha+j\delta} = \{x^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle x^\vee, \alpha \rangle = j\}.$$

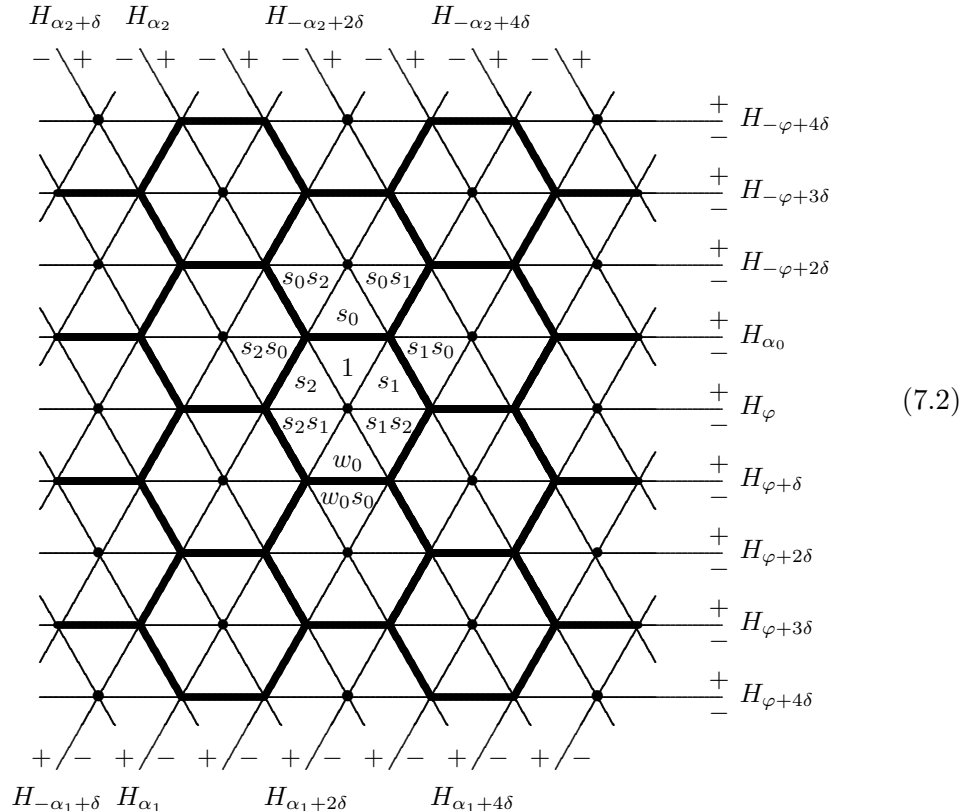
Under the map in (5.16) the chambers wC of the Tits cone X (see (2.20) and (2.21)) become the alcoves. Each alcove is a fundamental region for the action of W_{aff} on $\mathfrak{h}_\mathbb{R}$ given by (5.17) and W_{aff} acts simply transitively on the set of alcoves (see [Kac, Prop. 6.6]). Identify $1 \in W_{\text{aff}}$ with the *fundamental alcove*

$$A_0 = \{x^\vee \in \mathfrak{h}_\mathbb{R} \mid \langle x^\vee, \alpha_i \rangle > 0 \text{ for all } 0 \leq i \leq n\}$$

to make a bijection

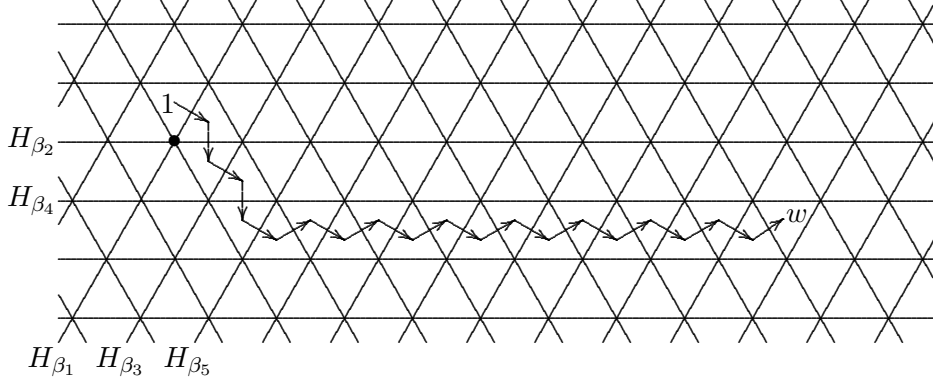
$$W_{\text{aff}} \longleftrightarrow \{\text{alcoves}\}.$$

For example, when $\mathfrak{g}_0 = \mathfrak{sl}_3$,



The alcoves are the triangles and the (centres of) hexagons are the elements of Q^\vee .

Let $w \in W_{\text{aff}}$. Following the discussion in (4.4)-(4.6), a reduced expression $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a *walk* starting at 1 and ending at w ,



and the points of

$$IwI = \{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}I \mid c_1, \dots, c_\ell \in \mathbb{C}\} \quad (7.3)$$

are in bijection with labelings of the edges of the walk by complex numbers c_1, \dots, c_ℓ . The elements of $R(w) = \{\beta_1, \dots, \beta_\ell\}$ are the elements of \tilde{R}_{re}^I corresponding to the sequence of hyperplanes crossed by the walk.

The labeling of the hyperplanes in (7.2) is such that neighboring alcoves have

$$v \begin{array}{c} H_{v\alpha_j} \\ | \\ \xrightarrow{vs_j} \end{array} \quad \text{with } v\alpha_j \in \tilde{R}_{\text{re}}^I \text{ if } v \text{ is closer to } 1 \text{ than } vs_j. \quad (7.4)$$

The *periodic orientation* (illustrated in (7.2)) is the orientation of the hyperplanes $H_{\alpha+k\delta}$ such that

- (a) 1 is on the positive side of H_α for $\alpha \in R_{\text{re}}^+$,
- (b) $H_{\alpha+k\delta}$ and H_α have parallel orientations.

This orientation is such that

$$v\alpha_j \in \tilde{R}_{\text{re}}^U \quad \text{if and only if} \quad v \begin{array}{c} H_{v\alpha_j} \\ | \\ \xrightarrow{+vs_j} \end{array}. \quad (7.5)$$

Together, (7.4) and (7.5) provide a powerful combinatorics for analyzing the intersections $U^-vI \cap IwI$. We shall use the first identity in (3.3), in the form

$$x_\alpha(c)n_\alpha^{-1} = x_{-\alpha}(c^{-1})x_\alpha(-c)h_{\alpha^\vee}(c) \quad (\text{main folding law}), \quad (7.6)$$

to rewrite the points of IwI given in (7.3) as elements of U^-vI . Suppose that

$$x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1} = x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v b, \quad \text{where } b \in I, \quad (7.7)$$

$v \in W_{\text{aff}}$ and $n_v = n_{j_1}^{-1} \cdots n_{j_k}^{-1}$ if $v = s_{i_1} \cdots s_{i_k}$ is a reduced word, and $\gamma_1, \dots, \gamma_\ell \in \tilde{R}_{\text{re}}^U$ so that $x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell) \in U^-$. Then the procedure described in (7.8)-(7.10) will compute $c'_{\ell+1} \in \mathbb{C}$, $b' \in I$, $v' \in W_{\text{aff}}$ and $\gamma_{\ell+1} \in \tilde{R}_{\text{re}}^U$ so that

$$x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}x_j(c)n_j^{-1} = x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)x_{\gamma_{\ell+1}}(c'_{\ell+1})n_{v'}b'.$$

Keep the notations in (7.7). Since $bx_j(c)n_j^{-1} \in Is_jI$ there are unique $\tilde{c} \in \mathbb{C}$ and $b' \in I$ such that $bx_j(c)n_j^{-1} = x_j(\tilde{c})n_j^{-1}b'$ and

$$\begin{aligned} x_{i_1}(c_1)n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}x_j(c)n_j^{-1} &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v bx_j(c)n_j^{-1} \\ &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_j(\tilde{c})n_j^{-1}b'. \end{aligned}$$

Case 1: If $v\alpha_j \in \tilde{R}_{\text{re}}^U$, $v \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xrightarrow{\tilde{c}} \\ + vs_j \end{array}$, then $x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_j(\tilde{c})n_j^{-1}b'$ is equal to

$$x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)x_{v\alpha_j}(\pm\tilde{c})n_{vs_j}b' \in U^-vs_jI \cap Iws_jI.$$

In this case, $\gamma_{\ell+1} = v\alpha_j$, $v' = vs_j$, and

$$v \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xrightarrow{\tilde{c}} \\ + vs_j \end{array} \quad \text{becomes} \quad v \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xrightarrow{\pm\tilde{c}} \\ + vs_j \end{array}. \quad (7.8)$$

Case 2: If $v\alpha_j \notin \tilde{R}_{\text{re}}^U$ and $\tilde{c} \neq 0$, $vs_j \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xleftarrow{\tilde{c}} \\ + v \end{array}$, then

$$\begin{aligned} x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_{\alpha_j}(\tilde{c})n_j^{-1}b' &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_{-\alpha_j}(\tilde{c}^{-1})x_{\alpha_j}(-\tilde{c})h_{\alpha_j^\vee}(\tilde{c})b' \\ &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_{-\alpha_j}(\tilde{c}^{-1})b'' \\ &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)x_{\gamma_{\ell+1}}(\pm\tilde{c}^{-1})n_v b'' \in U^-vI \cap Iws_jI, \end{aligned}$$

where $\gamma_{\ell+1} = -v\alpha_j$ and $b'' = x_{\alpha_j}(-\tilde{c})h_{\alpha_j^\vee}(\tilde{c})b'$. So

$$vs_j \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xleftarrow{\tilde{c}} \\ + v \end{array} \quad \text{becomes} \quad \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xrightarrow{\pm\tilde{c}^{-1}} \\ + v \end{array} \quad (7.9)$$

Case 3: If $v\alpha_j \notin \tilde{R}_{\text{re}}^U$ and $\tilde{c} = 0$, $vs_j \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xleftarrow{0} \\ + v \end{array}$, then

$$\begin{aligned} x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_{\alpha_j}(0)n_j^{-1}b' &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)n_v x_{-\alpha_j}(0)n_j^{-1}b' \\ &= x_{\gamma_1}(c'_1) \cdots x_{\gamma_\ell}(c'_\ell)x_{\gamma_{\ell+1}}(0)n_{vs_j}b' \in U^-vs_jI \cap Iws_jI, \end{aligned}$$

where $\gamma_{\ell+1} = -v\alpha_j$. So

$$vs_j \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xleftarrow{0} \\ + v \end{array} \quad \text{becomes} \quad vs_j \begin{array}{c} H_{v\alpha_j} \\ - | + \\ \xleftarrow{0} \\ + v \end{array} \quad (7.10)$$

We have proved the following theorem.

Theorem 7.1. *If $w \in W_{\text{aff}}$ and $\vec{w} = s_{i_1} \cdots s_{i_\ell}$ is a minimal length walk to w define*

$$\mathcal{P}(\vec{w})_v = \left\{ \begin{array}{l} \text{labeled folded paths } p \text{ of type } \vec{w} \\ \text{which end in } v \end{array} \right\} \quad \text{for } v \in W_{\text{aff}},$$

where a labeled folded path of type \vec{w} is a sequence of steps of the form

$$v \begin{array}{c} H_{v\alpha_j} \\ \leftarrow \begin{array}{|c} + \\ \hline c^{-1} \end{array} \rightarrow \\ \rightarrow \end{array} vs_j, \quad \begin{array}{c} H_{v\alpha_j} \\ \leftarrow \begin{array}{|c} + \\ \hline c^{-1} \end{array} \rightarrow \\ \rightarrow \end{array} v, \quad vs_j \begin{array}{c} H_{v\alpha_j} \\ \leftarrow \begin{array}{|c} + \\ \hline 0 \end{array} \rightarrow \\ \rightarrow \end{array} v, \quad \text{where the } k\text{th step has } j = i_k.$$

Viewing $U^-vI \cap IwI$ as a subset of G/I , there is a bijection

$$\mathcal{P}(\vec{w})_v \longleftrightarrow U^-vI \cap IwI.$$

8 An example

For the group $G = SL_3(\mathbb{C}((t)))$,

$$\begin{aligned} x_{\alpha_1}(c) &= \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & h_{\alpha_1^\vee}(c) &= \begin{pmatrix} c & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & n_1 &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ x_{\alpha_2}(c) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} & h_{\alpha_2^\vee}(c) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}, & n_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\ x_{\alpha_0}(c) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ct & 0 & 1 \end{pmatrix} & h_{\alpha_0^\vee}(c) &= \begin{pmatrix} c^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}, & n_0 &= \begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & 1 & 0 \\ t & 0 & 0 \end{pmatrix}. \end{aligned}$$

Let $w = s_2s_1s_0s_2s_0s_1s_0s_2s_0$ and $v = s_2s_1s_0s_2s_1s_2s_0$ so that

$$w = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t^{-2} & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 & 0 \\ t^2 & 0 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}.$$

We shall use Theorem 7.1 to show that the points of $IwI \cap U^-vI$ are

$$x_2(c_1)n_2^{-1}x_1(c_2)n_1^{-1}x_0(c_3)n_0^{-1}x_2(c_4)n_2^{-1}x_0(c_5)n_0^{-1}x_1(c_6)n_1^{-1}x_0(c_7)n_0^{-1}x_2(c_8)n_2^{-1}x_0(c_9)n_0^{-1}I,$$

with $c_1, \dots, c_9 \in \mathbb{C}$ such that

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad c_5 \neq 0, \quad c_6 = 0, \quad c_7 \neq 0, \quad c_9 = c_7^{-1}c_8. \quad (8.1)$$

Precisely,

$$x_2(0)n_2^{-1}x_1(0)n_1^{-1}x_0(0)n_0^{-1}x_2(0)n_2^{-1}x_0(c_5)n_0^{-1}x_1(0)n_1^{-1}x_0(c_7)n_0^{-1}x_2(c_8)n_2^{-1}x_0(c_7^{-1}c_8)n_0^{-1}$$

is equal to $u_9v_9b_9$, with $u_9 \in U^-$, $v_9 \in N$, $b_9 \in I$ given by

$$\begin{aligned} u_9 &= \begin{pmatrix} 1 & 0 & 0 \\ c_5^{-1} - c_5^{-2}c_7^{-1}c_8t & 1 & 0 \\ c_5^{-1}c_7^{-1}t^{-2} & 0 & 1 \end{pmatrix}, & v_9 &= \begin{pmatrix} 0 & 1 & 0 \\ -t^2 & 0 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \\ b_9 &= \begin{pmatrix} c_5^{-1} - c_5^{-2}c_7^{-1}c_8t & -c_5^{-2}c_7^{-1}c_8^2 & c_5^{-2}c_7^{-2}c_8^2 \\ -t^2 & c_5c_7 + c_8t & -c_5 - c_7^{-1}c_8t \\ -c_5^{-1}c_7^{-1}t^2 & -c_5^{-1}c_7^{-1}c_8t & c_7^{-1} + c_5^{-1}c_7^{-2}c_8t \end{pmatrix}, \end{aligned} \quad (8.2)$$

Step 5: If $c_5 \neq 0$ then by the folding law and the fact that $v_4 x_{-\alpha_0}(c_5^{-1}) v_4^{-1} = x_{-\alpha_1}(c_5^{-1})$,

$$u_4 v_4 b_4 x_0(c_5) n_0^{-1} = u_4 v_4 x_{-\alpha_0}(c_5^{-1}) x_{\alpha_0}(-c_5) h_{\alpha_0^\vee}(c_5) b_4 = u_4 x_{-\alpha_1}(c_5^{-1}) v_4 b_5 = u_5 v_5 b_5,$$

where

$$u_5 = u_4 x_{-\alpha_1}(c_5^{-1}), \quad v_5 = v_4, \quad \text{and} \quad b_5 = x_{\alpha_0}(-c_5) h_{\alpha_0^\vee}(c_5) b_4 = \begin{pmatrix} c_5^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & c_5 \end{pmatrix}.$$

Step 6: If $c_5^{-1} c_6 = 0$ (so $c_6 = 0$) then

$$u_5 v_5 b_5 x_1(c_6) n_1^{-1} = u_5 v_5 x_1(c_5^{-1} c_6) n_1^{-1} b'_5 = u_5 x_{-\alpha_2-2\delta}(0) v_5 n_1^{-1} b'_5 = u_6 v_6 b_6,$$

with

$$u_6 = u_5 x_{-\alpha_2-2\delta}(0), \quad v_6 = v_5 n_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -t & 0 \\ t^{-1} & 0 & 0 \end{pmatrix} \quad \text{and} \quad b_6 = b'_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_5^{-1} & 0 \\ -c_6 t & t & c_5 \end{pmatrix}$$

so that $b_5 x_1(c_6) n_1^{-1} = x_1(c_5^{-1} c_6) n_1^{-1} b'_5$.

Step 7: If $c_5 c_7 \neq 0$ then, since $v_6 x_{-\alpha_0}(c) v_6^{-1} = x_{-\varphi-2\delta}(c)$,

$$\begin{aligned} u_6 v_6 b_6 x_0(c_7) n_0^{-1} &= u_6 v_6 x_0(c_5 c_7) n_0^{-1} b'_6 = u_6 v_6 x_{-\alpha_0}(c_5^{-1} c_7^{-1}) x_{\alpha_0}(-c_5 c_7) h_{\alpha_0^\vee}(c_5 c_7) b'_6 \\ &= u_6 x_{-\varphi-2\delta}(c_5^{-1} c_7^{-1}) v_6 b_7 = u_7 v_7 b_7, \end{aligned}$$

where

$$\begin{aligned} u_7 &= u_6 x_{-\varphi-2\delta}(c_5^{-1} c_7^{-1}), \quad v_7 = v_6, \quad \text{and} \\ b'_6 &= \begin{pmatrix} c_5 & -1 & 0 \\ 0 & c_5^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b_7 = x_{\alpha_0}(-c_5 c_7) h_{\alpha_0^\vee}(c_5 c_7) b'_6 = \begin{pmatrix} c_7^{-1} & -c_5^{-1} c_7^{-1} & 0 \\ 0 & c_5^{-1} & 0 \\ -c_5 t & t & c_5 c_7 \end{pmatrix}, \end{aligned}$$

so that $b_6 x_0(c_7) n_0^{-1} = x_0(c_5 c_7) n_0^{-1} b'_6$.

Step 8: No restrictions on $c_5^{-2} c_7^{-1} c_8$. Since $v_7 x_{\alpha_2}(c) v_7^{-1} = x_{-\alpha_1+\delta}(-c)$,

$$u_7 v_7 b_7 x_2(c_8) n_2^{-1} = u_7 v_7 x_2(c_5^{-2} c_7^{-1} c_8) n_2^{-1} b'_7 = u_7 x_{-\alpha_1+\delta}(-c_5^{-2} c_7^{-1} c_8) v_7 n_2^{-1} b'_7 = u_8 v_8 b_8,$$

with

$$\begin{aligned} u_8 &= u_7 x_{-\alpha_1+\delta}(-c_5^{-2} c_7^{-1} c_8), \quad v_8 = v_7 n_2^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & t \\ t^{-1} & 0 & 0 \end{pmatrix}, \quad \text{and} \\ b_8 &= b'_7 = \begin{pmatrix} c_7^{-1} & -c_5^{-1} c_7^{-1} c_8 & c_5^{-1} c_7^{-1} \\ -c_5 t & c_5 c_7 + c_8 t & -t \\ -c_5^{-1} c_7^{-1} c_8 t & c_5^{-2} c_7^{-1} c_8^2 t & c_5^{-1} - c_5^{-2} c_7^{-1} c_8 t \end{pmatrix}, \end{aligned}$$

so that $b_7 x_2(c_8) n_2^{-1} = x_2(c_5^{-2} c_7^{-1} c_8) n_2^{-1} b'_7$.

Step 9: If $c_5^{-1} c_7 c_9 - c_5^{-1} c_8 = 0$ (so $c_9 = c_7^{-1} c_8$) then

$$u_8 v_8 b_8 x_0(c_9) n_0^{-1} = u_8 v_8 x_0(c_5^{-1} c_7 c_9 - c_5^{-1} c_8) n_0^{-1} b'_8 = u_8 x_{-\alpha_2-3\delta}(0) v_8 n_0^{-1} b'_8 = u_9 v_9 b_9$$

with u_9, v_9 and b_9 as in (8.2).

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