

141 Problems

JWR

Fall 2006

- Each problem has a point value as indicated. You may earn up to 500 points. (This represents half your final grade.)
- These problems require no specific knowledge besides arithmetic and high school algebra. The lectures correspond to the textbook, but the problems do not. My hope is that you will struggle with them and that that struggle will benefit you.
- You may collaborate on these problems, but if you do, you must disclose who you worked with. If you worked alone, so indicate. If you received help from someone, so indicate. If you used some written source in solving a problem, so indicate. In any case, the final writeup of a problem must be your own. Your grade will not depend on whether or not you worked alone or received help or used a written source.

THE FIRST PAGE OF ANY PROBLEM SET THAT YOU HAND IN MUST DISCLOSE WHO (IF ANYONE) YOU WORKED WITH, WHO (IF ANYONE) HELPED YOU, WHAT (IF ANY) OTHER SOURCES YOU USE, OR ELSE A DECLARATION THAT YOU WORKED ALONE.

The reason for requiring disclosure is to reinforce academic honesty as a value.¹

- For all these problems, it is not enough to simply state an answer; you must show your work in such a way that someone unfamiliar with this course (but reasonably skilled at arithmetic and high school algebra) can follow your reasoning. Your grade will be based in part on the clarity of the exposition.
- You may ask me for help either by email, in class, or during office hours.
- Show respect for the grader. Messy work will be marked down.
- You may resubmit any problem. The problem will be graded as if it were the first submission, and the final grade on the problem will be the grade on the final submission.

¹ Mathematicians frequently discuss their research with each other, but always acknowledge these discussions in the final product.

- In order to insure that you do not put off doing any problems until late in the semester the following deadlines are imposed. To earn the full 500 points you must
 1. earn 150 points on work handed in before Monday October 2,
 2. earn 300 points on work handed in before Monday October 30, and
 3. earn all 500 points on work handed in before Monday November 27.

If you fail to meet these deadlines, your score will be reduced. For example, if you only earn 140 points on work handed in before Monday October 2 you will lose 10 points.

1. (20 points) Taxes. A company has 1000 employees each earning \$50,000 per year and paying 10% in income taxes. The CEO pays 30% of his income in taxes and pays as much in taxes as all of his employees put together. What is his income?

2. (20 points) Francs and Dollars. A sign at the O'Hare airport currency exchange says

We buy Swiss francs for .7473, sell for .8954

At Zurich Airport the sign at the currency exchange says

We buy dollars for 1.1775, sell for 1.2775

1. Should a traveller from Chicago to Zurich change money at O'Hare or at Zurich? How much does the wrong choice cost? What about a traveller from Zurich to Chicago?
2. Alice buys Swiss francs in Chicago and immediately sells them back. How much did she lose?
3. Bob buys dollars in Chicago and immediately sells them back. How much did he lose?
4. Charles buys Swiss francs in Zurich and immediately sells them back. How much did he lose?
5. Diane buys dollars in Zurich and immediately sells them back. How much did she lose?

3. (5 points) What Percent? Assume that the population of the city of Madison is 200,000 and that the population of the city of Milwaukee is 600,000. Complete the following statements:

- (a) The population of Madison is ...% of the population of Milwaukee.
- (b) The population of Milwaukee is ...% of the population of Madison.

- (c) The population of Madison is ...% less than the population of Milwaukee.
- (d) The population of Milwaukee is ...% more than the population of Madison.

You may round your answer to the nearest whole number percent. (For instance, 16.6% percent could be expressed as 17%.)

4. (20 points) **Eggs.** If a hen and a half lays an egg and a half in a day and a half. how many eggs will five hens lay in seven days?
5. (20 points) **Filling a pool.** If a wide hose fills a swimming pool in one hour and a narrow hose fills the pool in an hour and a half, how long will it take for both hoses together to fill the pool?
6. (20 points) **Shoes.** A shoe salesman once told me I shouldn't wear the new shoes he was selling me every day, or even wear them every other day: "You can double the lifetime of your shoes by wearing them only every third day," he told me confidently. Give two different interpretations of what the salesman meant; explain why, under the first interpretation, his advice was good advice, and why, under the other interpretation, his advice was bad advice.
7. (30 points) **William and Cecille.** A short obituary in the December 5, 2005 issue of The New Yorker (p. 40) read as follows:

A letter arrived the other day bearing the news that Cecille Shawn, the wife of William Shawn, who was the editor of The New Yorker from 1952 to 1987, had passed away. The letter was from one of the Shawn's three children, the composer Allen Shawn, and he wrote to say that "our mother died peacefully at home on October 30th. She was ninety-nine years old." Cecille Shawn, who was born Cecille Lyon, grew up in Chicago and, as a young woman, worked as the editor of the features page of the Chicago Daily News. She created a game using a character named Blundering Ben, in which readers scored points by finding mis-used words and phrases. When she was eighteen, Cecille was introduced to William Shawn, and he wrote in his diary, "Today I met my future wife." They married in 1928 and spent much of the next year living in Paris, where Mr. Shawn earned a living playing jazz piano in a club called La Cloche. It was their only trip overseas. When they returned to the States, the Shawns moved from Chicago to New York. The Depression was on, but Cecille was able to find a job as a freelance "fact-finder" at The New Yorker, which was in the eighth year of its existence. She asked the editors if her husband could help with the work. As Allen wrote, "She soon happily bowed out of her own job at the magazine," where her husband thrived for the next half-century.

For how many years did William court Cecille before marrying her? Express your answer as a range, in the form "More than X years but less than Y years". Make the range as narrow as possible, given the information you have, but no narrower than logic permits you to conclude.

8. (15 points) **Business Trips.** Roger and Rhonda plan to have a child together. Her menstrual cycle typically lasts 25 to 29 days, and ovulation occurs on day 12, 13, or 14 of her cycle. Rhonda wants Roger to be at home during each cycle on the day of her ovulation. If Rhonda's last cycle began on July 22 (Day 1), should Roger plan to take a business trip in the last half of September? If he absolutely needs to take the trip, what are the days in late September when he must make sure to be at home?

9. (15 points) **Shampoo Coupons.** Suppose a supermarket that sells 10-ounce bottles of shampoo for \$1.00 apiece and 15-ounce bottles of shampoo for \$1.20 apiece offers 60-cents-off coupons (and no other kind), which are good for either size of shampoo; and suppose you somehow wind up with more of these coupons than you can ever use, so that you'll never run out. In order to save money, should you buy 10-ounce bottles of the shampoo or 15-ounce bottles? Explain. (Note: You can only use one coupon per bottle of shampoo.)

10. (15 points) **Head Starts.** George and Martha had a race. By the time George reached the finish line, Martha was already 5 meters beyond the finish line. (Assume that she kept running at the same speed for the whole time that George was running, even after she'd won.) They decide to race again, but this time Martha gives George a head-start by letting him start the race 5 meters past the starting line. Assuming that both runners run at the same pace as before, who will win? (Is there enough information to solve this problem? Notice that I did not specify the length of the race in meters. Of course it is more than 5 meters, or else the problem would not make sense.)

11. (25 points) **Coffee and Milk.** I had mug containing 1 cup of pure coffee and a jug containing 2 cups of pure milk. I took some teaspoonfuls from the mug, put them into the jug, and stirred. Then I took an equal number of teaspoonfuls from the jug, put them into the mug, and stirred. So now the mug contains 1 cup of liquid that's part coffee and part milk, and the jug contains 2 cups of liquid that's part milk and part coffee. Which are true?

- (1) There is more coffee in the jug than milk in the mug.
- (2) There is less coffee in the jug than milk in the mug.
- (3) There are equal amounts of coffee in the jug and milk in the mug.
- (4) It is impossible to determine which of (1)-(3) is true without more information.
- (5) There is more coffee in the mug than milk in the jug.
- (6) There is less coffee in the mug than milk in the jug.
- (7) There are equal amounts of coffee in the mug and milk in the jug.
- (8) it is impossible to determine which of (5)-(7) is true without more information.

(As usual, you must give reasons to receive any credit.)

12. (15 points) **Generous Tipppers.** I go to a restaurant to eat a \$10 meal. I see a generous friend of mine. If I join my friend, who plans to eat an \$8 meal, she'll generously split the bill with me. However, my friend is also a generous tipper: if I eat with her, we'll pay a 20% tip, whereas on my own I tend to pay only 15%. Which option will save me money: eating alone, or eating with my friend? And how much money will I save? (For purposes of this problem, ignore sales tax.)

13. (15 points) **My Salary.** (a) If my salary increases by 10% one year and then increases by 10% in the next year, what is the net increase over the 2-year period? (b) If my salary decreases by 10% one year and then decreases by 10% in the next year, what is the net decrease over the 2-year period? (c) If my salary increases by 10% one year and then decreases by 10% in the next year, what is the net percentage change over the 2-year period? (d) If my salary decreases by 10% one year and then increases by 10% in the next year, what is the net percentage change over the 2-year period?

14. (15 points) **Paying Debt in one Day.** The national debt is about 8 trillion dollars. The typical American household's net worth is about 45 thousand dollars. If we decreased the national debt by the typical American household's net worth every second, roughly how long would it take to pay down the national debt? (Ignore the fact that the economy is constantly changing.) Express your answer in appropriate units. That is, don't leave it as some huge number of seconds or minutes or days; use units like weeks, months, or years, so that the number is smaller and the answer is easier to make intuitive sense of.

15. (15 points) **Elections and Football Fields.** The 2000 election would (probably) have ended in a win for Al Gore if 500 Florida voters who voted for Bush or Buchanan had voted for Gore. The number of Americans who voted in 2000 was roughly 100 million. If, in order to get a more intuitive sense of how narrow a margin of victory this was, we represent the number of voters in the U.S. by the length of a football field (100 yards), how narrow was Bush's margin of victory, in inches?

16. (15 points) **Life and 141.** Say you devote 10 hours a week on average to Math 141 for 15 weeks (that includes attending lectures, doing the reading, doing homework, and taking tests). What percent of your life (assuming you live 75 years) will have been devoted to this class? If we represent your lifetime by one day, by how much time should we represent the portion of your life spent on Math 141? (Use appropriate units. That is, if your answer is less than one hour, don't use units of hours; and if your answer is less than one minute, don't use units of minutes.)

17. (20 points) **The Consumer Price Index.** Say that the CPI for Tampa, FL is 168.5 this year and the CPI for New York, NY this year is 212.7 (and assume that both CPI's are expected to remain unchanged in the short run).

1. If I have to choose between a job in Tampa that pays \$30,000 per year and a job in New York City that pays \$40,000 per year, which job will give me more buying power, as estimated by that cities' respective consumer price indices?
2. Suppose the salary offer for both jobs goes up by \$20,000, so that the Tampa job pays \$50,000 per year and the New York job pays \$60,000 per year. Which job will give me more buying power?

18. (20 points) Varia. The land of Varia is divided into two equally populace provinces, Lovaria and Hivaria. In each of the two provinces, the average height of adult males is 5 ft. 8 inches. However, in Lovaria the standard deviation is 2 inches, while in Hivaria the standard deviation is 4 inches. Assume that in each province, the distribution of heights follows a normal distribution.

1. Among those adult male inhabitants of Varia who are 6 feet tall or taller, what percentage come from Lovaria?
2. Among those adult male inhabitants of Varia who are 6 feet 2 inches tall or taller, what percentage come from Lovaria?
3. In light of what you learned from parts (a) and (b), explain why the following reasoning does or does not make sense: "Although Africans constitute less than 50% of the world's population, over half of the people over 7 feet tall are African; therefore, Africans are taller on average than non-Africans." (Leave aside the issue of whether the speaker's numerical assertions — "less than 50%", "over half" — are correct, and focus on the "therefore".)

19. (25 points) Testing for Antibodies. Two researchers, Smith and Jones, conduct research on the presence of a certain antibody in the population.

	Test Positive?	
	Yes	No
Female	3	97
Male	1	49

Smith's study

	Test Positive?	
	Yes	No
Female	5	45
Male	9	91

Jones' study

1. Which sex has the higher percentage of positive tests in the Smith study?
2. Which sex has the higher percentage of positive tests in the Jones study?
3. Which sex has the higher percentage of positive tests in the both studies combined?

20. (40 points) Car Rental. Your company has started to do business in West Texas. Representatives will rent a car at the Lubbock airport in order to make their calls on clients in the area. Your boss has told you to find out which car

rental agency will be cheapest for a one-day rental. There are three car rental agencies, and their rates for a one-day trip are as follows:

COMPANY	BASE RATE	MILEAGE RATE
Acme	\$12	28 cents per mile
Beavis	\$20	21 cents per mile
Criterion	\$26	12 cents per mile

(The total charge is the sum of the base rate and the mileage cost for the trip.)

1. What should you report to your boss?
2. Later, Beavis has offered your company a price concession: on a one day trip, the first 30 miles will be free. Does this affect their competitive position?

21. (40 points) **The Moneylenders.** In Plastic City there is a heavy demand for small loans. Smiley and Sharkey are the two moneylenders who meet this demand. They both have a minimum loan amount of \$120, with a loan period of one year. Their terms for repayment are as follows:

Smiley: pay \$12 at the end of each month for 12 months.

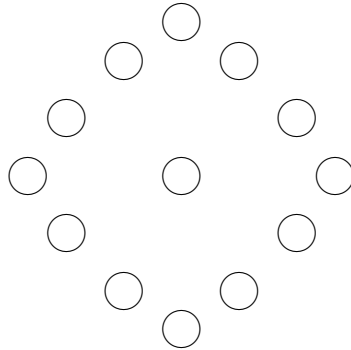
Sharkey: pay \$150 at the end of the 12th month.

Both are serious, professional moneylenders: as soon as they get \$120 of repayments, they use it to make a new loan.

You are discussing loans with your friends, and one of them says “Well, it’s obvious who to borrow from.” A second says “Borrowing doesn’t get you anywhere, what we should do is become moneylenders ourselves. If only we had some money to lend.” A third friend thinks for a while, then says “Actually, we could do it. We could take out loans from one of those guys and use what we borrowed to make loans the way one of those guys does, more or less, and wind up money ahead. I figure that if we took out ten \$120 loans now, and refinanced twice, we could take in enough to pay off what we borrowed at the end of the third year and still have money over.”

What could your friend mean? Who would you borrow from? Whose terms would you lend on? How would you both repay and have money over by the end of the third year?

22. (40 points) **Fortuna.** You’re lining up your summer job plans, and you have an idea: you’ll set up a booth at Wisconsin Dells, charging people to play a game you’ve thought up. You’re playing a candidate game to test it. It’s played with 13 billiard balls (numbered 1-13) and a board with 13 ball-shaped depressions as pictured:



The balls are shuffled and one of them is randomly placed in the center. The player has 30 seconds to place the remaining 12 balls in the surrounding depressions in such a way that the sum of the 3 balls along any straight line through the center is the same. If the player succeeds, she wins a number of dollars equal to the row sum.

1. Is every game winnable?
2. How much money can a player win in a single game?
3. How much should you charge a person to play the game? Remember, you aim to make money off this venture, but no one will play if you charge too much.

Hint: Which numbers, when placed in the center, make the game winnable? How can you be certain that these are the only numbers that work?

23. (40 points) **Gold Bricks** You are the Real Outside Gold Brick Company. The bricks you make are cubes. To make a cube, you first make a framework of soda straws, Then you cover the framework with squares of gold foil (like aluminum foil, but made of gold) Gold foil is .01 inches thick and costs the same per pound as gold in any other form. You can sell your bricks for one-twentieth the cost of a solid gold cube of the same dimensions. The money you get for your first brick will buy enough gold foil to cover nine bricks of the same dimensions as the first brick.

1. Assess the following options for using the new foil: a) make nine bricks of the original dimensions b) make three bricks, each with side three times the original c) make one big brick Will one bring in more money than another?
2. How big was your original brick?
3. Suppose you will be able to repeat the cycle:

make new brick(s), sell them

a total of fifteen times before pocketing the profits. Each time, you make the new bricks in the most profitable size. How big will be the last brick(s) you sell?

24. (20 points) **New Harmony.** Four students came together a year ago and founded the New Harmony Friendship Club. They decided they would start their meetings by having each member shake hands with each other member, and wrote a by-law to this effect. At the time, this involved a total of 6 handshakes, and got the meetings off to a harmonious start. Now club membership has grown to 10, and the handshake ceremony takes a while. As club President, you would like to see membership grow to 50 or even 100. How many handshakes are involved with the present 10 members? How many would be involved for a large number of members (like 100)? What are your reasons why this must be the correct answer? .

If a single handshake takes 5 seconds, what is the least possible time in which a four person club could accomplish its handshakes? Demonstrate how they would do this. What about an five person club? An eight person club? Again, demonstrate how they would do this.

25. (40 points) **The Lifetime Ball of Twine.** I bought this just three years ago. It was a BIG ball (not really a ball, see the sketch), ADDSKETCH and I joked that it was my lifetime supply. (The twine unwinds from central hole.) Now I am older and achier, and the hole in the center of the ball has grown to 3 inches diameter, and I am uneasy about my joke. If it wasn't a joke, and if I keep on using the twine at the same rate, how long do I have to live?

I use the twine to tie up newspapers for recycling. I have been making the stacks 6 inches high. Suppose I make them 9 inches high; will this increase my hypothetical life expectancy? By how much? (Suppose continue to tie the same way and that the papers continue to come in at the same rate.)

26. (40 points) **How You Get Salt.** You find a stretch of level desert next the ocean and bulldoze a shallow basin in it. You let salty seawater run into the basin until it is full, then shut the valve and go away to a cooler place while the desert sun evaporates the water. After this has happened, you return and get the salt.

1. You run 100,000 pounds of salty water into your basin. It contains 2% salt. You come back a week later and find that the solution in your basin is now 3% salt. Assuming that the same number of pounds of water evaporate each day, what % salt will you find if you come back one week later yet? How do you know your answer is right?
2. When is the soonest you can come back and find solid salt? (Days count; you are paying interest on a loan to build your basin, and need to raise cash by selling the salt).
3. You could install reflectors that would increase by 20% the amount of water evaporated each day. If you did that, what would be the total time from start to solid salt? How can you tell your answer is right?

- Someone tells you they know a better way to get salt. Each morning, open the valve and let in enough new salt water to refill your basin to its original level. That way you should wind up with a lot more salt when the basin finally dries out. Is this a more efficient way to get salt or a crazy notion? Why? Recall that you want to wind up with solid salt. Your efficiency is

Total amount of salt obtained / total days required to obtain it.

27. (40 points) **Time To Weigh the Hippos.** Martha is the chief hippopotamus caretaker at the Wild Animal Park in San Diego, California. She has just arrived at the cargo dock in the downtown harbor in order to pick up four members of the endangered species hippopotamus mathematicus recently rescued from African poachers. In order to complete the paperwork, she needs to weigh them, but the only scale big enough to weigh a hippo is a truck scale that starts at 350 kilograms - more than any of the hippos weighs. Martha is puzzled for a few minutes, then gets the idea of weighing them in pairs, thinking that if she gets the weight of every possible pair, she can later figure out the weights of the individual hippos. She measures the weights pair by pair, getting 362, 516, 406, 428 and 494. As she tries to weigh the heaviest pair of hippos, the scale breaks.

- What is the weight of the lightest pair of hippos?
- What are the weights of all the individual hippos?
- Explain why your answer to question (2) tells the truth, the whole truth, and nothing but the truth about the weights of the individual hippos.

28. (40 points) **A Bad Bike Day.** Joan lives 7 miles down a straight road from school. Last Tuesday she rode her bike home from school. Here's what happened.

She needed to be home by 5:00 to catch a ride to the mall. She left school at 4:00, and started pedaling home at a steady 10 miles per hour. After going 3 miles, she realized she had left her purse next the bike rack at school. She pedaled back, hurrying and making it back to school in 12 minutes. Her purse was there and she again started home, keeping up her new faster pace. She pedaled for 20 minutes, when suddenly her bike got a flat tire. She fussed with the bike for 6 minutes, then abandoned it and ran the rest of the way home at 8 miles per hour.

- Draw a graph of Joan's velocity against time for the trip. Velocity is positive when going toward home, negative when going back toward school.
- Draw a graph of her distance from school, also against time.
- Explain how you can get one of these graphs from the other without referring back to the story.
- Did Joan make it home in time to catch her ride?

Figure 1: Historical Estimates of World Population

Year	Est. Pop.	Year	Est. Pop.	Year	Est. Pop.	Year	Est. Pop.
-200	200	800	224	1800	954	1960	3,000
1	250	1000	254	1850	1,241	1970	3,700
200	260	1200	400	1900	1,633	1980	4,500
400	206	1400	374	1925	2,000	1990	5,300
600	206	1600	579	1850	2,555	2000	6,100

(Population in millions.)

29. (40 points) Swisconsin Tax. The State of Swisconsin (Simplified Wisconsin) gets 5% every time money is spent. If a dollar is spent in a store, Swisconsin gets 5 cents of it as sales tax, and the merchant gets 95 cents. If the dollar is paid in wages, Swisconsin gets 5 cents in income tax, and the employee gets 95 cents. All transactions are done with perfect accuracy (no rounding to the nearest cent).

1. If an employer pays a dollar to a worker, and the worker spends what remains after taxes in the grocery, what does the grocer get after taxes? Why?
2. And if the grocer spends the above amount at the wholesaler, how much does the wholesaler get after taxes? Is there a pattern to your calculations so far?
3. How much does Swisconsin get on the transactions listed above? As the remnants of that starting dollar continue to be spent and re-spent forever, what is the ultimate total that Swisconsin will get from it? Why?
4. What does your answer to part 3 imply for the ultimate distribution of money in Swisconsin? Why? Could your answer fit with what you see in real-life Wisconsin? If yes, how? If no, how does the real state avoid the effects seen in the simplified model?

30. (40 points) The Family Tree Game. Your family tree consists of your parents and their parents, and their parents' parents, etc. In this game every person has exactly two parents, who are exactly 25 years older than the person him/herself. This is an approximation which produces simple calculations and doesn't get too far from reality.

Your family tree comes in levels spaced 25 years apart. The first level, 25 years back, consists of your parents; the second level, 50 years back consists of your grandparents, etc. Just counting the size of the levels leads to some interesting questions.

1. How many people are in the 100-year back level of your family tree? In the 200 year back level? Based on the numbers, what are the chances of a

randomly selected person from 1799 being part of your family tree? (See the population table² in Figure 30.)

2. Kublai Khan led the Golden Horde to conquer large parts of Europe and Asia. He died in 1294. What do the numbers say about the chances of his being in your family tree? (Again, see the population table.)
3. The population tables are careful estimates. Why does it become increasingly difficult, as you go back in time, to reconcile them with the calculations you have done? What do you think is wrong with the assumptions?
4. Societies make rules to prevent lines of ancestry or descent from merging too rapidly. The most common rule, and the one we assume here, is that it is forbidden to marry brothers, sisters or first cousins. Brothers and sisters are people with whom you share a parent. First cousins are people with whom you share a grandparent (thus, the children of your parents brothers and sisters). Under these rules, what is the smallest number of grandparents you can have? The smallest number of great-grandparents? Of great-great grandparents? Why?

31. (40 points) Poison and Soap. *Poison* is a game played with two players and ten pennies. A move consists of taking either one or two pennies off the desk. At each move, the player decides whether to take one or two pennies this time. Player move in turn. The Player that takes the last (poison) penny loses.

1. Would you rather move first or second? Why?
2. If you move first, is there a strategy you can follow to ensure that you will win? If so, what is it?
3. What if you move second?
4. If you start the game with 210 pennies instead of 10, is it better to move first or to move second? Why?

Soap is a game played with a special chocolate bar. Like many chocolate bars, this one is marked with lines, and you can break off pieces along any line (you have to break the whole line). What is special about the bar is that the little square in the upper left hand corner is soap, not chocolate, and the player who gets it loses, and has to eat it.

1. Would you rather play first or second? Why?
2. Is there a strategy by which the first player can be sure to win? If so, what is it? Why does it have to work?

² Don't take this table too seriously, especially the pre twentieth century estimates. I chose the information from various websites and they disagree.

3. Same question about a strategy by which the second player can be sure to win.
4. Starting with a bigger chocolate bar, 5 squares by 5 squares, is it better to move first or second? What strategy would guarantee a win? What are the reasons why this strategy has to work?
5. What about starting with bar 4 squares by 5 squares?
6. The championship version of Soap is played with a big bar, 10 squares by 15 squares. In the championship game, is it better to move first or second? What strategy will guarantee a win? What are the reasons why this strategy has to work?

32. (25 points) **Cruising Down the River.** Tom and Huck went down the Mississippi River on a steamboat in 8 days. Then they returned by the same boat, taking 12 days. Then they drifted down the river on a raft. How long did the raft trip take them?

33. (40 points) **Boats.** The ferry boats Alfie and Belle are based on opposite sides of Lake Tana, Alfie on the west side and Belle on the east. They start running at 8:00 a.m. each morning. Each boat goes to the base on the opposite side, spends an hour unloading and loading, then returns to its home base. On the first trip, the captain of the Alfie can see, when the boats pass each other, that he is 8 miles from the eastern shore. The second time they pass, he can see he is 4 miles from the western shore. (Each boat goes at its own constant speed throughout its trip.) What is the width of Lake Tana? Why?

This is a peculiar problem. There are five unspecified quantities: the width of the lake, the speed of Alfie, the speed of Belle, the time when the ferries pass the first time, and the time when they pass the second time. However, there are only four relations among these quantities. There should be infinitely many solutions for the five unspecified quantities but, because the two ferries spend the same time unloading and loading, all these solutions have the same value for the width. To solve this problem first find the combined distance the two ferries have traveled when they first pass, and then the combined distance the two ferries have traveled the second time they pass.

34. (40 points) **Bet the Farm.** Here are two separate claims about positive integers. Would you bet the farm on the truth of either of them?

1. No matter what integer you start with, the sum of three consecutive integers is always a multiple of 3.
2. No matter what integer you start with, the sum of the squares of three consecutive integers is never a multiple of 3.

A conservative person might not want to bet on any claim which covers so much ground – not just numbers like 1 and 2, or even 10 and 11 but numbers so big you would be at a loss even to write them down. But there are statements, like “the square of an even number is always even” that are true for all integers, and

that you know are true because there is a reason for them to be true. On the other hand there are statements which are true for small integers, but not for all integers. For example,

$$n^2 - n + 41 \text{ is a prime for } n = 1, 2, \dots, 40$$

(I wrote a computer program to check this) but

$$n^2 - n + 41 \text{ is not a prime for } n = 41$$

(it is divisible by 41).

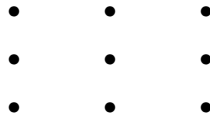
35. (30 points) Milk. Joan is making a recipe that calls for 2 tablespoons of whole milk. However, she does not have whole milk; all she has is skim milk and half-and-half. Perusing the cartons, she finds that a 1-cup serving of skim milk contains 0 g fat and a 2-tbsp serving of half-and-half contains 3 g fat (there are 16 tablespoons to a cup). Rummaging through the garbage, Joan finds the recently emptied and discarded whole-milk carton, which tells her that a 1-cup serving of whole milk contains 8 g fat. Using only a teaspoon (= 1/3 of a tablespoon), how can Joan combine suitable amounts of skim milk and half-and-half to form a liquid with the same fat content as whole milk?

36. (30 points) IRS Meets 141. The tax laws encourage certain kinds of financial behavior by allowing tax deductions for them. One example is setting up an IRA (Individual Retirement Account) –that is, putting money in an account where it is held until you reach retirement age, when you can withdraw it. Each year you may be able to deduct the amount of your IRA contribution from your taxable income, thus reducing your tax that year. If you can deduct a contribution to an IRA, then you do not pay tax that year on the part of your income used to fund the contribution. But that bit of tax is not forgiven, only postponed until you withdraw the money from your IRA account. Also postponed until that time is the tax on interest earned on your IRA contribution. Which of these two postponements is the more valuable in the long run? More precisely, assume you have just earned \$2000 and have two choices:

- (A) Pay tax on the \$2000 now, deposit what remains after tax in an account, let the interest accumulate without tax for 30 years and then pay tax on the total interest but not the original deposit.
- (B) Deposit the entire \$2000 now; each year pay tax out of the account on that year's interest, and let the remaining interest accumulate. At the end of 30 years, pay tax on your original deposit only.

Which choice leaves you with the largest after-tax sum 30 years hence? Assume that you will always be in the 28% tax bracket, and that your contribution always earns 10% per year, compounded annually. Does your answer depend on these assumptions?

37. (5 points) **Thinking outside the box.** Draw four straight lines through the nine dots without lifting your pencil.



38. (5 points) **I_{max}.** Make four equilateral triangles out of six toothpicks of the same length.

39. (30 points) **The Lambada Turnpike.** When you enter the Lambada Turnpike, they give you a ticket showing the time and place of your entry. When you exit, you turn in this ticket and they use it to figure your toll. Since they know the distance between toll stations, they can also use it to check your average speed against the turnpike limit of 65 mph. On your trip, heavy snow limits your speed to 40 mph for the first 120 miles. How fast can you drive for the remaining 300 miles without having your ticket prove that you broke the speed limit?³

40. (30 points) **Watermelons.** A load of watermelons is delivered to market. It weighs 1000 pounds, of which 98% is water. While the load is waiting to be sold, some of the water evaporates so that when the load is sold, it is only 96% water. What is the weight of the load at the time it is sold?

41. (30 points) **Jack and Jill.** Jack and Jill run on a circular track of radius 300 feet. Jack starts at noon and runs clockwise at 600 feet per minute. Jill starts at the same point one minute later and runs counterclockwise at 800 feet per minute. When will they first pass one another?

42. (30 points) **The Final Exam.** A math instructor wants to give a final exam so that by the end of the semester s/he has examined all parts of the course equally. The first 33 topics have already been examined for three hours, i.e. at 1/11 hour per topic. The last 6 topics have not yet been examined. The final exam will be two hours in length and will contain questions from the whole course. What portion of the final exam should be dedicated to the last 6 topics in order that each of the 39 topics be examined equally in the 5 hours of exams?

³Regardless of the answer, don't speed.