

Math 221 – Exam I – Friday October 6, 2000

Answers

I. (25 points.) Let $y = \frac{\sin x}{\cos x}$. Find dy/dx and d^2y/dx^2 .

Answer:

$$\frac{dy}{dx} = \frac{\sin'(x) \cos(x) - \cos'(x) \sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = (\cos x)^{-2}$$

so

$$\frac{d^2y}{dx^2} = -2(\cos x)^{-3}(-\sin x) = \frac{2 \sin x}{\cos^3 x}.$$

Grader's Comments. *In regards to students' performance on Problem I of the exam, the big mistake was just not knowing the Quotient Rule. That's a shame, because I thought it was an easy 25 points.*

II. (25 points.) If a point traces the circle $x^2 + y^2 = 25$ and if $dx/dt = 7$ when the point reaches $(3, 4)$ find dy/dt there.

Answer: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$ so $2 \cdot 3 \cdot 7 + 2 \cdot 4 \cdot \left. \frac{dy}{dt} \right|_{(x,y)=(3,4)} = 0$ so $\left. \frac{dy}{dt} \right|_{(x,y)=(3,4)} = -\frac{21}{4}$.

III. (25 points.) Two functions f and g are *inverse functions* if the equations $y = f(x)$ and $x = g(y)$ have the same graph, i.e. if $x = g(f(x))$ and $y = f(g(y))$. What is the inverse function g of the function $f(x) = x^3$ and what is the derivative of g ?

Answer: The inverse function is $g(y) = y^{\frac{1}{3}}$ so

$$g'(y) = \frac{y^{-\frac{2}{3}}}{3}.$$

This is a consequence of the Chain Rule: $g(f(x)) = x$ so $g'(f(x))f'(x) = 1$ so if $y = f(x)$, then

$$g'(y) = g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{3x^2} = \frac{1}{3y^{\frac{2}{3}}}.$$

Grader's Comments. *More than half of the students did my question – find the inverse of $f(x) = x^3$ and its derivative – correctly. For the others, about half claimed the inverse as x^{-3} , another half as $x = y^3$. The notation – whether to use x or y as the variable of the inverse function – is part of the reason the students got confused. We'd better make a clear rule on it to the students.*

Remark. *The function g defined by the formula*

$$g(y) = y^{\frac{1}{3}}$$

is the same as the function g defined by the formula

$$g(x) = x^{\frac{1}{3}}.$$

IV. (25 points.) State and prove the formula for the derivative of the quotient of two functions. In your proof you may use (without proof) the limit laws, the theorem that a differentiable

Answer: Suppose that $f(x) = u(x)/v(x)$ for all x where u and v are differentiable and $v(a) \neq 0$. Then

$$\begin{aligned}
 f'(a) &= \\
 &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} && \text{(definition)} \\
 &= \lim_{x \rightarrow a} \frac{(u(x)/v(x)) - (u(a)/v(a))}{x - a} && \text{(hypothesis)} \\
 &= \lim_{x \rightarrow a} \frac{u(x)v(a) - u(a)v(x)}{v(x)v(a)(x - a)} && \text{(hsa)} \\
 &= \lim_{x \rightarrow a} \left(\frac{u(x) - u(a)}{x - a} \cdot \frac{v(a)}{v(x)v(a)} - \frac{u(a)}{v(x)v(a)} \cdot \frac{v(x) - v(a)}{x - a} \right) && \text{(hsa)} \\
 &= \left(\lim_{x \rightarrow a} \frac{u(x) - u(a)}{x - a} \right) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot \left(\lim_{x \rightarrow a} \frac{v(x) - v(a)}{x - a} \right) && \text{(lim law)} \\
 &= u'(a) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot v'(a) && \text{(definition)} \\
 &= \frac{u'(a)v(a) - u(a)v'(a)}{v(a)^2} && \text{(hsa)}
 \end{aligned}$$

(In the fifth step the theorem that a differentiable function is continuous is also used.)

Grader's Comments. *The most common mistake I saw was the incorrect use of free and dummy variables as in*

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

I also saw the expression $f'(u/v)$ where $(u/v)'$ would have been better. Another frequent error was mixing x and a as in

$$f'(a) = \frac{u'(a)v(x) - u(a)v'(a)}{v(a)^2}$$

These errors tell me that the students do not know how to use the notation correctly.

V. (25 points.) The function

$$f(x) = x^{1/2} + x^{-1/2}$$

is to be approximated near $x = 4$ by a quadratic function

$$Q(x) = C_0 + C_1(x - 4) + C_2(x - 4)^2$$

by choosing coefficients C_0, C_1, C_2 in such a way that

$$f(4) = Q(4), \quad f'(4) = Q'(4), \quad f''(4) = Q''(4).$$

Find the coefficients.

Answer: $Q(4) = C_0$, $Q'(4) = C_1$, and $Q''(4) = 2C_2$ while $f(4) = 2\frac{1}{2}$, $f'(4) = \frac{1}{2}4^{-\frac{1}{2}} - \frac{1}{2}4^{-\frac{3}{2}} = \frac{3}{16}$, and $f''(4) = -\frac{1}{4}4^{-\frac{3}{2}} + \frac{3}{4}4^{-\frac{5}{2}} = -\frac{1}{16}$. Hence

Grader's Comments. Basically, those who knew how to do the problem got it right and only made a few minor algebraic mistakes those who got it wrong had mainly these problems: 1) not knowing how to differentiate with $C_0, C_1, C_2 \dots$ didn't treat them as constants, 2) algebra (hsa) for exponents...ie, not knowing that $4^{-1/2} = 1/2$. I mostly saw $(1/4) \cdot (4)^{-1/2} = 1 ???$...however, those who left it in more complicated terms and choose not to simplify did not make those mistakes and got full credit, so...the moral is?

VI. (25 points.) Find the limit. Distinguish between an infinite limit and one which doesn't exist. (Give reasons!)

(a) $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2}$

Answer: By a change of the dummy variable $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = \lim_{u \rightarrow 0^+} \frac{\sin u}{u} = 1$.

(b) $\lim_{x \rightarrow \infty} \frac{\sin(x^2)}{x^2}$

Answer: Since $-\frac{1}{x^2} \leq \frac{\sin(x^2)}{x^2} \leq \frac{1}{x^2}$, the limit is zero by the Sandwich Theorem.

(c) $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right)$

Answer: $\sin\left(\frac{1}{x^2}\right) = 1$ if $x = 1/\sqrt{2n\pi + \pi/2}$ and $\sin\left(\frac{1}{x^2}\right) = -1$ if $x = 1/\sqrt{2n\pi - \pi/2}$ where n is an integer. Hence as $x \rightarrow 0$ the value $\sin\left(\frac{1}{x^2}\right)$ oscillates between -1 and 1 and so the limit does not exist.

(d) $\lim_{x \rightarrow 0} \frac{\sin 1}{x^2}$

Answer: $\lim_{x \rightarrow 0} \frac{\sin 1}{x^2} = (\sin 1) \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$. (Note that $\sin 1 > 0$ since $0 < 1 < \pi$.)

(e) $\lim_{h \rightarrow 0} \frac{\sin(a+h) - \sin a}{h}$

Answer: By the definition of the derivative this is $\sin'(a)$, the derivative of the sine function evaluated at a . In other words,

$$\lim_{h \rightarrow 0} \frac{\sin(a+h) - \sin a}{h} = \cos a.$$

Grader's Comments. On the limit problems, most people seemed to be guessing. Many people said that if you have zero in the denominator, then the limit DNE. Also many people believed that $\lim(\sin x/x) = 1$, regardless of whether $x \rightarrow 0$ or not. The most egregious mistakes however were things like :

$$\sin(a+h) = \sin a + \sin h, \quad \sin(x^2) = (\sin x) \cdot x, \quad \sin 1 = 0.$$