Math 221 – Exam II – Tuesday Evening October 31, 2000
Answers

I. (25 points.) (1) State the Mean Value Theorem.

Answer: If $f(x)$ is continuous on $a \le x \le b$ and differentiable on $a < x < b$, then there is a point c with $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(The geometric interpretation is that the tangent line to the graph at $(c, f(c)$ is parallel to the "secant line" joining $(a, f(a))$ and $(b, f(b))$.)

(2) Let f be a function, a a number in its domain, and n a nonnegative integer. Define the degree n Taylor polynomial of f centered at a.

Answer: It is the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}.$$

It is the unique polynomial of degree n which has the same derivatives as f at a up to order n, i.e.

$$P_n^{(k)}(a) = f^{(k)}(a) \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

(3) The hypothesis of the Extended Mean Value Theorem is that that $f(x)$ is $n + 1$ times differentiable and that $f^{(n+1)}$ is continuous on an interval, that a and b are two numbers in that interval, and that $P_n(x)$ is the degree n Taylor polynomial of $f(x)$ centered at a. State the rest of the Extended Mean Value Theorem.

Answer: There is a number c_{n+1} between a and b such that

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c_{n+1})(b-a)^{n+1}}{(n+1)!}.$$

Grader's Comments: After grading I sat back and reflected. Students did fairly well stating the MVT but not the EMVT, which has more symbols. The definition of the degree nTaylor polynomial was alright except I saw f^k

instead of $f^{(k)}$ a lot. Our students have trouble writing sentences. They get confused between hypotheses and conclusions and go ungrammatical at the slightest provocation.

II. (25 points.) Evaluate the following limits. If the limit does not exists write DNE and say why. Distinguish between a limit which is infinite and one which does not exist. EXPLAIN YOUR REASONING.

(i) $\lim_{x\to 0} \frac{\sqrt{1+x}-1-(x/2)}{x^2}$.

Answer: By l'Hôpital's Rule twice:

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1-(x/2)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x} = \lim_{x\to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\frac{1}{8}.$$

Both numerator and denominator of the first two limits vanish when evaluated at 0 so l'Hôpital's Rule applies. (This is Example 3 on page 163.)

(ii) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Answer:

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0} \frac{x-\sin x}{x\sin x} && \text{(HSA)} \\
&= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} && (0/0\ ) \\
&= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} && \text{(l'Hôpital)} \\
&= 0. && \text{(0/2 plug in)}
\end{aligned}$$

(This is Example 7 page 165.)

Grader's Comments: Those who understood how to use l'Hopital's rule correctly got the 1st limit correct. Most mistakes here were not remembering the power rule in differentiating $(1+x)^{1/2}$ and bad algebra mistakes. I found a few people who differentiated $x/2$ using the quotient rule?! The last most common mistake was not recognizing the numerator goes to zero, since simplifying $1^(1/2) - 1 + 0 =$ (many strange answers).

In the second limit, MANY people did not get this correct. the biggest problem was they wrote garbage like $1/0 - 1/0 = 0$ not knowing they needed to get a common denominator and proceed with l'Hopital's rule. Here also, if they did do the first step correctly, many got points off for not using the product rule when differentiating $x \sin x$.

My thoughts are 1) they forgot simple differentiation techniques and 2) they are scared of exponents.

III. (25 points.) A ladder 26 ft long leans against a vertical wall. The foot of the ladder is being drawn away from the wall at a rate of 4 ft/sec. How fast is the top of the ladder sliding down the wall at the instant when the foot of the ladder is 10 ft from the wall?

Answer: Step 1. Name everything in sight:

> Let x= distance from foot of ladder to wall, y= height of the top of the ladder above the ground, and t= time in seconds.

Step 2. Express the relations in the problem in this notation:

$$\frac{dx}{dt} = 4, \qquad x^2 + y^2 = 26^2.$$

Step 3. Write what we are asked to find:

$$\left.\frac{dy}{dt}\right|_{x=10} = ?$$

Step 4. Compute:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

so

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{4x}{y}$$

so

$$\left.\frac{dy}{dt}\right|_{x=10} = -\frac{40}{\sqrt{26^2 - 10^2}} = -\frac{40}{24} = -\frac{5}{3}.$$

(This is Example 2 on page 139.)

Grader's Comments: More than half students did this related rates problem correctly. The most common mistake was claiming that the derivative with respect to time of the constant 26^2 is 2*26=52 rather than 0.

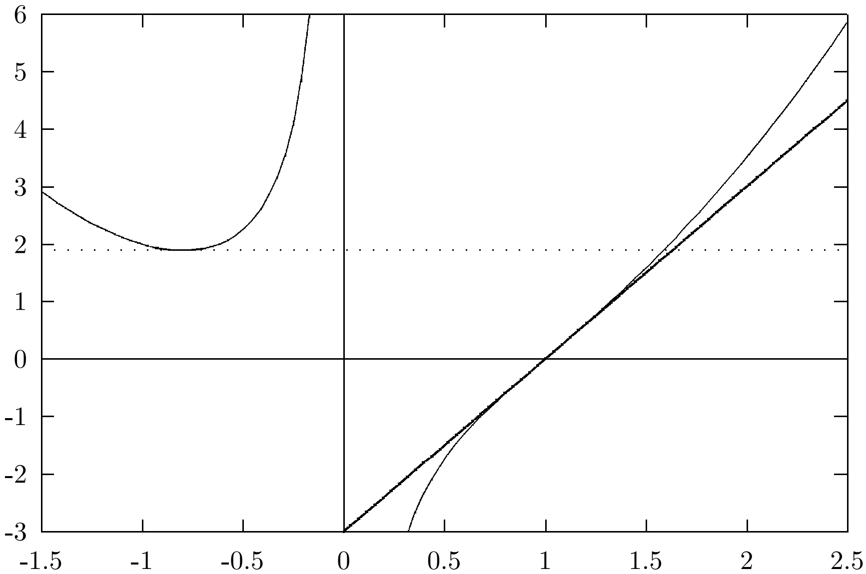
IV. (25 points.) Graph $y = x^2 - x^{-1}$. Determine the intervals where y is increasing (as x increases), and where the graph is concave up. Indicate the turning points and points of inflection on the graph and draw the tangent line at each.

Answer: We find the important points.

$$\lim_{x \to -\infty} y = \infty, \qquad \lim_{x \to 0-} y = \infty, \qquad \lim_{x \to 0+} y = -\infty, \qquad \lim_{x \to \infty} y = \infty.$$

The derivative $y' = 2x + x^{-2}$ vanishes at $x = -(\frac{1}{2})^{1/3}$. The second derivative $y'' = 2 - 2x^{-3}$ vanishes at $x = 1$. In the table denote $(\frac{1}{2})^{1/3}$ by a for readibility.

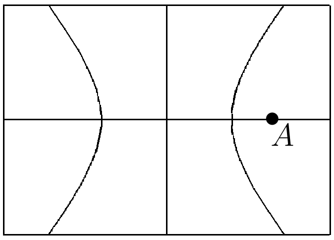
x	y	y'	y''	
$-\infty$	∞			
$x < -a$		$-$	$+$	$\searrow \cup$
$-a$		0		low turning point
$-a < x < 0$		$+$	$+$	$\nearrow \cup$
$0-$	∞			
$0+$	$-\infty$			
$0 < x < 1$		$+$	$-$	$\nearrow \cap$
1			0	inflection point
$1 < x$		$+$	$+$	$\nearrow \cup$
∞	∞			



Grader's Comments: Many students got the problem IV (the graph) correct - it is after all very similar to example 2 on page 136 of the test. A few students found the intervals $f' > 0$ and $f'' > 0$ but then drew graphs which which did not correspond to what they had found. By far the most common mistake was to ignore the vertical asympotote at the y axis. Algebra was also a common stumbling block - many students did not know how to solve the equation $x + x^{-2} = 0$ - they would factor out x and then argue that $x = 0$ is a solution.

Thomas Finney V is (as I now realize) deficient in its treatment of graphing in that it suggests the student plug in a few values (as in Table 3-2) page 137. This deemphasizes the importance of the vertical asymptote and the use of the Intermediate Value Theorem to calculate the intervals where $f' > 0$ and $f'' > 0$.

V. (25 points.) Find the points on the curve $y^2 = x^2 - 1$ that are nearest the point $A(a, 0)$ in case (1) $a = \sqrt{8}$ and (2) $a = \sqrt{2}$. Suggestion: The algebra is simpler if you minimize the square of the distance rather than the distance. Do not plug in for a till the end of the problem so you can do both parts at the same time.



Answer: The points $P(x,y)$ on the curve with $x \le -1$ are far from A so we restrict attention to those with $x \ge 1$. Let w be the square of the distance from the point $P(x,y)$ on the curve to the point A. Then

$$w = (x-a)^2 + (y-0)^2 = (x-a)^2 + x^2 - 1 = 2x^2 - 2ax + a^2.$$

Hence

$$\frac{dw}{dx} = 4x - 2a$$

which vanishes when $x = a/2$, i.e. When $(x,y) = \left(\frac{a}{2}, \pm\sqrt{\frac{a^2}{4} - 1}\right)$. As

$$\frac{d^2w}{dx^2} = 4 > 0$$

such a point must be a minimum. If $a = \sqrt{8}$ the points are $(x,y) = (\sqrt{2}, \pm 1)$, but if $a = \sqrt{2}$ there is no such point as the quantity under the square root sign is negative. In the latter case the distance (which is not defined for $-1 < x < 1$) is increasing so the closest point is the endpoint $P(1,0)$.

Grader's Comments: Nobody got a perfect score on my problem! Many people just didn't even understand what they were supposed to do at all. For example, some people just plugged in the values of a for x, or tried to use the Mean Value Theorem somehow.

Of those who did attempt to find the minimum distance, many people were hampered by not using the right formula. In most cases this led to severe problems because the algebra became unmanageable. Also, some people tried to take the derivative without first eliminating the y variable. This would be ok if they had used the chain rule, but the majority did not.

Finally, only one person got the second part right. Perhaps in the future we need to do more examples in which the answer is not at the critical point, but at an endpoint.

VI. (25 points.) (i) Find the polynomial of degree two which best approximates the function $f(x) = 7 + 5x^2 + x^4$ near $x = 1$.

Answer: It is the degree 2 Taylor polynomial of f centered at 1:

$$P(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2}.$$

$f(1) = 13$, $f'(x) = 10x + 4x^3$ so $f'(1) = 14$ and $f''(x) = 10 + 12x^2$ so $f''(1) = 22$. Hence

$$P(x) = 13 + 14(x-1) + 11(x-1)^2.$$

(ii) For $f(x)$ and $P(x)$ as in part (i) prove an inequality of form

$$|f(x) - P(x)| \leq M|x-1|^3$$

valid in the range $1 < x < 2$. (You are suppose to find a number M which makes the inequality true for all x in the interval $1 < x < 2$.)

Answer: By the Extended Mean Value Theorem for each x there is a c between 1 and x such that

$$f(x) - P(x) = \frac{f'''(c)(x-1)^3}{6}.$$

The third derivative is $f'''(x) = 24x$ so $0 < f'''(c) < 48$ if $1 < c < x < 2$. Hence

$$|f(x) - P(x)| \leq \frac{48|x-1|^3}{6}$$

for $1 < x < 2$.

Grader's Comments: 1. The first part (computing the Taylor Polynomial) was done OK by most of the examinees. However some students failed to realize that the in the coefficients one has to substitute 1 for x, i.e., there terms were: $f^(n+1)(x)(x-a)^(n+1)/(n+1)!$. This abomination was penalized by taking off 8 points. Apart from this, there were a huge number of the usual algebra mistakes, the most intriguing being the repeated occurence of $(1)^4 = 4$. On the whole the impression was that they understood what Taylor polynomials are and how to compute them.

2. Very few people got the second part correct. A large majority of them understood that they needed to use the Extended Mean Value Theorem. A smaller percentage could compute the various terms in the remainder term. However the real problem was how to estimate the unknown quantity c occurring in the formula. Here there were several routes that they took:

: Set $c = 1$.

: Set $c = 2$, without explanation.

: Give correct explanation of the fact that $c = 2$.

Another peculiar phenomenon was that the students replaced x by particular values of that variable.(Frequently 2, 1.5 or 1.1).

In general, the performance on the second question was much less than satisfactory.Thebasic problem it seemed was with the undetermined constant c in the formulas. Perhaps we should have spent more time with Rolle's and Mean Value Theorems where these concepts occur in a simpler and more geometric setting than in the Extended Mean Value theorem.

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There are  199 scores
grade      range       count      percent
   A     125...150     16           8.0%
  AB     120...124      7           3.5%
   B     100...119     47          23.6%
  BC      80... 99     75          37.7%
   C      65... 79     30          15.1%
   D      60... 64     10           5.0%
   F       0... 59     14           7.0%
Mean score =  91.5. Mean grade = 2.45.
;-- Question averages:
;--18.005   13.2915   22.005   14.8241   6.53266   16.799

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