I. (30 Points.) Find the indicated derivative.

(a) \( y = \ln(1 + x^2) \). \( \frac{dy}{dx} = ? \)

Answer:

\[
\frac{dy}{dx} = \frac{2x}{1 + x^2}.
\]

(b) \( f(x) = e^{-x^2/2} \). \( f''(x) = ? \) (Note: 2nd derivative.)

Answer:

\[
f'(x) = -xe^{-x^2/2} \quad f''(x) = -e^{-x^2/2} + x^2e^{-x^2/2}.
\]

II. (30 Points.) (a) Find the equation of the tangent line to the graph

\( x + y + e^y = 3 + e \)

at the point \((x, y) = (2, 1)\).

Answer: The slope of the tangent line is

\[
m = \left. \frac{dy}{dx} \right|_{(x,y) = (2,1)}
\]

By implicit differentiation we have that

\[
1 + \frac{dy}{dx} + e^y \frac{dy}{dx} = 0
\]

at any point of the curve. Evaluating at \((x, y) = (2, 1)\) gives

\[
1 + m + em = 0, \quad \text{so} \quad m = -\frac{1}{1 + e}.
\]

The tangent line is the line through \((1, 2)\) with slope \(m\); its equation is

\[
y = 1 + m(x - 2), \quad \text{i.e.} \quad y = 1 - \frac{x - 2}{1 + e}.
\]

(b) The function \( y = f(x) \) is implicitly defined by the above equation, i.e.

\[
x + f(x) + e^{f(x)} = 3 + e.
\]

Use linear approximation to find \( f(2.1) \) approximately. Answer: The graph of the linear approximation \( y = L(x) \) is tangent line. The value at \( x = 2.1 \) is

\[
L(2.1) = 1 - \frac{2.1 - 2}{1 + e} = 1 - \frac{0.1}{1 + e}.
\]

III. (25 Points.) The count in a bacteria culture grows exponentially. If the count is 200 after two hours and 600 after six hours, when will the count be 900?

Answer: The general formula for the bacteria count \( N(t) \) at time \( t \) is

\[
N(t) = N_0e^{kt}
\]

where \( N_0 \) is the count when \( t = 0 \). We are given that \( N(2) = 200 \) and \( N(6) = 600 \) and we must find a value of \( t \) for which \( N(t) = 900 \).

\[
200 = N_0e^{2k}, \quad 600 = N_0e^{6k}, \quad 900 = N_0e^{kt}.
\]
Divide the second equation by the first to get \(3 = e^{4k}\) so \(k = \ln(3)/4\). Divide the third equation by the second to get \(3/2 = e^{k(t-6)}\). Hence \(\ln(3/2) = kt - 6k\) so

\[
t = \frac{\ln(3/2)}{k} + 6 = \frac{4(\ln(3/2))}{\ln(3)/4} + 6 = \frac{16(\ln(3/2))}{\ln(3)} + 6 = \frac{22}{\ln(2)}.
\]

IV. (25 Points.) State and prove a formula for the derivative \(f'(x)\) of the function

\[f(x) = 2^x.
\]

Justify each step. You may use without proof the fact that the limit

\[
\ln 2 = \lim_{h \to 0} \frac{2^h - 1}{h}
\]

exists. You should not use (without proof) the formulas for the derivative of \(e^x\) or \(\ln x\). Answer:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{definition of } f'(x)
\]

\[
= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \quad \text{definition of } f(x)
\]

\[
= \lim_{h \to 0} \frac{2^x \cdot 2^h - 2^x}{h} \quad \text{high school algebra}
\]

\[
= \lim_{h \to 0} \frac{2^x(2^h - 1)}{h} \quad \text{high school algebra}
\]

\[
= \frac{2^x}{h} \lim_{h \to 0} \frac{2^h - 1}{h} \quad \text{limit law}
\]

\[
= 2^x \ln 2
\]

(\bigstar)

V. (30 Points.) (a) \(\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = ?\)

Answer: \(\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1.\)

(b) \(\lim_{x \to \infty} (\cosh(x))^2 - (\sinh(x))^2 = ?\) Hint: \(\cosh(x) = \frac{e^x + e^{-x}}{2}\) and \(\sinh(x) = \frac{e^x - e^{-x}}{2}\)

Answer: \((\cosh(x))^2 - (\sinh(x))^2 = 1\) for all \(x\); hence the limit is 1.

(c) \(\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} = ?\)

Answer:

\[
\ln \left[ \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} \right] = \lim_{x \to \infty} \ln \left(1 + \frac{2}{x}\right)^{3x} = \lim_{x \to \infty} 3x \ln \left(1 + \frac{2}{x}\right) = \lim_{x \to \infty} 3 \ln \left(1 + \frac{2}{x}\right).\]

By l’Hôpital’s rule:

\[
\lim_{x \to \infty} 3 \ln \left(1 + \frac{2}{x}\right) = \lim_{x \to \infty} 3 \frac{(1 + 2x^{-1})^{-1} \cdot (-2x^{-2})}{-2x^{-2}} = \lim_{x \to \infty} 6 \left(1 + \frac{2}{x}\right)^{-1} = 6.
\]

Hence

\[
\ln \left[ \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} \right] = 6
\]

so

\[
\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} = e^6.
\]
VI. (30 Points.) I am observing a meteor through a telescope. The meteor is falling straight downward toward an impact point 30 miles away from me. When the meteor is 40 miles above the ground, it is falling at a speed of 1 mile per second. At that moment, how fast must I be changing the angle my telescope makes with the horizontal in order to keep the meteor in view?

Answer: Let $y$ be the height of the meteor above the ground and $\theta$ be the angle the telescope makes with the ground. Then $\theta = \tan^{-1}(y/30)$. Then

$$\frac{d\theta}{dt} \bigg|_{y=40} = \frac{1}{1 + (y/30)^2} \cdot \frac{1}{30} \frac{dy}{dt} \bigg|_{y=40} = \frac{1}{1 + (40/30)^2} \cdot \frac{1}{30} \cdot (-1) = -\frac{30}{50^2} = -\frac{3}{250}.$$

VII. (15 Points.) Suppose that $f$ and $g$ are inverse functions, that $f'(1) = 5$, $f'(2) = 7$, $f'(3) = 9$, $f(1) = 2$, $f(2) = 3$ and $f(3) = 4$. Find $g'(2)$. Justify your answer. If there is insufficient information to do this problem, say why.

Answer: $f(g(y)) = y$ so $f'(g(y))g'(y) = 1$ by the chain rule, so $g'(y) = 1/f'(g(y))$ so $g'(2) = 1/f'(g(2))$. But $f(1) = 2$ so $g(2) = 1$. Hence $g'(2) = 1/f'(1) = 1/5$.

VIII. (15 Points.) Let $F(x) = x \tan^{-1}(1/x)$ if $x \neq 0$ and $F(0) = 0$.

(a) Is $F$ continuous at 0? (Justify your answer.)

Answer: 

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x \tan^{-1}(1/x) = \lim_{x \to 0^+} x \lim_{y \to \infty} \tan^{-1}(y) = 0 \pi = 0 = F(0)$$

and similarly

$$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} x \tan^{-1}(1/x) = \lim_{x \to 0^-} x \lim_{y \to -\infty} \tan^{-1}(y) = 0(-\pi) = 0 = F(0)$$

so $\lim_{x \to 0} F(x) = F(0)$ and therefore $F$ is continuous.

(b) Is $F$ differentiable at 0? If so, what is $F'(0)$? (Justify your answer.)

Answer: 

$$\lim_{x \to 0^+} \frac{F(x) - F(0)}{x - 0} = \lim_{x \to 0^+} \tan^{-1}(1/x) = \lim_{y \to \infty} \tan^{-1}(y) = \pi$$

and similarly

$$\lim_{x \to 0^-} \frac{F(x) - F(0)}{x - 0} = \lim_{x \to 0^-} \tan^{-1}(1/x) = \lim_{y \to -\infty} \tan^{-1}(y) = -\pi$$

so the derivative $F'(0)$ does not exist.