Taylor’s Formula

(The Extended Mean Value Theorem)

October 19, 2000

§1 When \( f \) is a function and \( k \geq 0 \) is an integer the notation \( f^{(k)} \) denotes \( k \)th derivative of \( f \). Thus

\[
f^{(0)}(x) = f(x), \quad f^{(1)}(x) = f'(x), \quad f^{(2)}(x) = f''(x),
\]

and so on. Given a number \( a \) in the domain of \( f \) and an integer \( n \geq 0 \), the polynomial

\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}
\]

is called the degree \( n \) Taylor polynomial of \( f \) centered at \( a \). The Taylor polynomial \( P_n(x) \) is the unique polynomial of degree \( n \) which has the same derivatives as \( f \) at \( a \) up to order \( n \):

\[
P_n^{(k)}(a) = f^{(k)}(a) \quad \text{for } k = 0, 1, 2, \ldots, n.
\]

§2 The letter \( \sum \) is the Greek \( S \) (for sum) and is pronounced \( \text{sigma} \) so the notation used in \((\#)\) is called \textbf{sigma notation}. It is a handy notation but if you don’t like it you can indicate the summation with dots:

\[
\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.
\]

Hence the first few Taylor polynomials are

\[
P_0(x) = f(a),
\]

\[
P_1(x) = f(a) + f'(a)(x-a),
\]

\[
P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2},
\]

\[
P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}.
\]

§3 The Taylor polynomial \( P_n(x) \) for \( f(x) \) centered at \( a \) is the polynomial of degree \( n \) which best approximates \( f(x) \) for \( x \) near \( a \). The precise statement is
**Taylor’s Theorem.** Suppose that $f$ is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous. Let $a$ be a point in the domain of $f$. Then

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0. \quad (∗)$$

§ 4 In order to use Taylor’s formula approximate a function $f$ we pick a point $a$ where the value of $f$ and of its derivatives is known exactly. Then the Taylor polynomial $P_n(x)$ can be evaluated exactly for any $x$. We then need to “estimate the error” $f(x) - P_n(x)$, i.e. to find an inequality

$$|f(x) - P_n(x)| \leq M|x - a|^{n+1}$$

which tells us how small the error $f(x) - P_n(x)$ is, i.e. how close $P_n(x)$ is to $f(x)$. A value for $M$ is usually found via

§ 5 The Extended Mean Value Theorem. Suppose that $f$ is $n+1$ times differentiable and that $f^{(n+1)}$ is continuous on an interval, let $a$ and $b$ be two numbers in that interval, and let $P(x)$ be the Taylor polynomial of $f$ centered at $a$. Let $b$ be a point in the domain of $f$. Then for each $b$ there is a number $c_{n+1}$ between $a$ and $b$ such that

$$f(b) - P_n(b) = \frac{f^{(n+1)}(c_{n+1})(b-a)^{n+1}}{(n+1)!}.$$  

§ 6 Note that the formula for the error $f(b) - P_n(b)$ is the same as the next term in the series (∗) except that the $n + 1$st derivative $f^{(n+1)}$ is evaluated at the unknown point $c_{n+1}$ instead of $a$. The Extended Mean Value Theorem is proved in problem 74 on page 174 of the text. Equation (∗) is an immediate consequence.

§ 7 Exercise. Evaluate $\sum_{k=3}^{5} \frac{1}{k}$.

§ 8 Exercise. Let $f(x) = \sqrt{x}$. Find the polynomial $P(x)$ of degree three such that $P^{(k)}(4) = f^{(k)}(4)$ for $k = 0, 1, 2, 3$.

§ 9 Exercise. Let $f(x) = x^{1/3}$. Find the polynomial $P(x)$ of degree two which best approximates $f(x)$ near $x = 8$.

§ 10 Exercise. Let $f(x)$ and $P(x)$ be as in § 9. Evaluate $P(10)$ and use the Extended Mean Value Theorem to prove that

$$|10^{1/3} - P(10)| \leq \frac{10}{6 \cdot 27 \cdot 32}.$$  

Hint: The function $g(x) = x^{-8/3}$ is decreasing so $g(10) < g(8)$.

§ 11 Exercise. Find a polynomial $P(x)$ of degree three such that

$$\lim_{x \to 0} \frac{\sin(x) - P(x)}{x^3} = 0.$$
Use the Extended Mean Value Theorem to show that

\[ |\sin(x) - P(x)| \leq \frac{|x|^4}{24}. \]