I. (60 points.) (a) Find $x = x(t)$ if $\frac{dx}{dt} + \frac{x}{1 + t^2} = 0$ and $x(0) = x_0$.

**Answer:** By separation of variables

\[
\ln |x| = \int \frac{dx}{x} = - \int \frac{dt}{1 + t^2} = - \tan^{-1}(t) + C
\]

so (as $|x| = \pm x$) we get

\[
x = \pm \exp(- \tan^{-1}(t) + C) = c \exp(- \tan^{-1}(t)), \quad c = \pm e^C.
\]

As $\tan^{-1}(0) = 0$ the initial condition gives $x_0 = c$. so

\[
x = x_0 \exp(- \tan^{-1}(t)).
\]

(b) Find $y = y(t)$ if $\frac{dy}{dt} + \frac{y}{1 + t^2} = \exp(- \tan^{-1} t)$ and $y(0) = y_0$.

**Answer:** This is an inhomogeneous linear equation and the general solution

\[
x(t) = c\Phi(t), \quad \Phi(t) := \exp(- \tan^{-1} t).
\]

of the corresponding homogeneous linear equation was found in part (a). We use as Ansatz

\[
y(t) = c(t)\Phi(t).
\]

Then

\[
\frac{dy}{dt} + \frac{y}{1 + t^2} = \left(c'(t)\Phi(t) + c(t)\Phi'(t)\right) + \frac{c(t)\Phi(t)}{1 + t^2}
\]

\[
= c'(t)\Phi(t) + c(t) \left( \Phi'(t) + \frac{\Phi(t)}{1 + t^2} \right)
\]

\[
= c'(t)\Phi(t)
\]

\[
= \exp(- \tan^{-1} t) \quad \text{if } c'(t) = 1
\]

i.e. if $c(t) = t + \text{constant}$. Evaluating at $t = 0$ shows that the constant must be $y_0$ so $c(t) = t + y_0$ so the solution is

\[
y(t) = (t + y_0) \exp(- \tan^{-1} t).
\]
II. (40 points.) (a) State the Existence and Uniqueness Theorem for Ordinary Differential Equations.

Answer: If $f(t, x)$ has continuous partial derivatives then the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

has a unique solution $y = y(x)$.

(b) State the Exactness Criterion for the differential equation

$$M(x, y) \, dx + N(x, y) \, dy = 0.$$  

Answer: If $M(x, y)$ and $N(x, y)$ have continuous partial derivatives, then there is a function $F = F(x, y)$ solving the equations

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(c) Does the differential equation

$$(1 + x^2 + x^4y^6) \, dx + (1 + x^2 + y^2) \, dy = 0$$

have a solution $y = y(x)$ satisfying the initial condition $y(0) = 5$? (Explain your answer.)

Answer: Yes. The differential equation can be written as

$$\frac{dy}{dx} = f(x, y), \quad f(x, y) = -\frac{1 + x^2 + x^4y^6}{1 + x^2 + y^2}$$

so the Existence and Uniqueness Theorem applies. (The Exactness Criterion is irrelevant here.)

III. (30 points.) Consider the differential equation $\frac{dx}{dt} = (1 - x)x(1 + x)$. 

2
(a) Draw a phase diagram.

**Answer:**

![Phase Diagram]

(b) Determine the limit \( \lim_{t \to \infty} x(t) \) if \( x(t) \) is the solution with \( x(0) = -0.5 \).

**Answer:** From the phase diagram \( \lim_{t \to \infty} x(t) = -1 \).

(b) Determine the limit \( \lim_{t \to \infty} x(t) \) if \( x(t) \) is the solution with \( x(0) = -1 \).

**Answer:** The constant function \( x(t) = -1 \) satisfies both the differential equation and initial condition \( x(0) = -1 \). Therefore it is the only solution by the the Existence and Uniqueness Theorem so \( \lim_{t \to \infty} x(t) = \lim_{t \to \infty} -1 = -1 \). Similarly if \( x(0) = 0 \) then \( \lim_{t \to \infty} x(t) = \lim_{t \to \infty} 0 = 0 \).

IV. (60 points.) A 1200 gallon tank initially holds 900 gallons of salt water with a concentration of 0.5 pounds of salt per gallon. Salt water with a concentration of 11 pounds of salt per gallon flows into the tank at a rate of 8 gallons per minute and the well stirred mixture flows out of the tank at a rate of 3 gallons per minute. Write a differential equation for the amount \( x = x(t) \) of salt in the tank after \( t \) minutes. (You need not solve the differential equation but do give the initial condition.) **SHOW YOUR REASONING.**

**Answer:** After \( t \) minutes \( 8t \) gallons of saltwater has flowed into the tank and \( 3t \) gallons has flowed out so the volume of the saltwater in the tank is \( V = 900 + 5t \). The concentration of salt in this saltwater is \( x/V \) pounds per gallon. In a tiny time interval of size \( dt \) the amount of saltwater flowing out of the tank is \( 3 \) \( dt \) gallons and the amount of salt in this saltwater is \( (x/V) \times (3 \) \( dt \) \) pounds. In this same tiny time interval the amount of saltwater flowing in is \( 8 \) \( dt \) gallons and the amount of
salt in that saltwater is \( 11 \times 8 \, dt = 88 \, dt \) pounds. Hence the net change in the amount of salt is

\[
dx = 88 \, dt - \frac{3x \, dt}{V} = \left( 88 - \frac{3x}{900 + 5t} \right) \, dt.
\]

Initially \( V = 900 \) gallons so \( x(0) = 0.5 \times 900 \) pounds. Thus the ODE is

\[
\frac{dx}{dt} = 88 - \frac{3x}{900 + 5t}, \quad x(0) = 450.
\]

The fact that the tank holds 1200 gallons means that it is full after 60 minutes. This problem (with different numbers) is Example 5 on page 53 of the text.

**V. (60 points.)** A projectile is launched straight upward from its initial position \( y_0 \) with initial velocity \( v_0 > 0 \). Air resistance exerts a force proportional to the square of the projectile’s velocity so that Newton’s second law gives that

\[
\frac{dv}{dt} = \frac{F_G + F_R}{m} = -1 - v|v|.
\]

(To simplify the problem we chose units where the gravitation constant is \( g = 1 \).) The projectile goes up for \( 0 \leq t < T \) and then goes down.

(a) Find a formula for \( v(t) \) for \( 0 < t < T \).

**Answer:**

\[
\tan^{-1} v = \int \frac{dv}{1 + v^2} = - \int dt = -t + \tan^{-1} v_0
\]

so

\[
v = \tan(-t + \tan^{-1} v_0) = \frac{v_0 - \tan t}{1 + v_0 \tan t}.
\]

(b) Find a formula for \( v(t) \) for \( t > T \).

**Answer:**

\[
\tanh^{-1} v = \int \frac{dv}{1 - v^2} = \int dt = -t + T
\]

as \( v = 0 \) when \( t = T \) so

\[
v = \tanh(T - t).
\]

(c) Find a formula \( T \).

**Answer:** From part (a) \( v_0 - \tan T = 0 \) so \( T = \tan^{-1}(v_0) \).
### Table of Integrals and Identities

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<thead>
<tr>
<th>Integral</th>
<th>Identity</th>
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<tr>
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<td>( \frac{a + u}{a - u} = e^{2aw} \iff \frac{u}{a} = \frac{e^{aw} - e^{-aw}}{e^{aw} + e^{-aw}} )</td>
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<tr>
<td>( \int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left</td>
<td>\frac{u + a}{u - a} \right</td>
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<tr>
<td>( \sin(t) = \frac{e^{it} - e^{-it}}{2i} )</td>
<td>( \sinh(t) = \frac{e^t - e^{-t}}{2} )</td>
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<td>( \cos(t) = \frac{e^{it} + e^{-it}}{2} )</td>
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<td>( i \tan(t) = \frac{e^{it} - e^{-it}}{e^{it} + e^{-it}} )</td>
<td>( \tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} )</td>
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<td>( \cos^2(t) + \sin^2(t) = 1 )</td>
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<td>( d \sin(t) = \cos(t) , dt )</td>
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</tr>
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<td>( d \cos(t) = -\sin(t) , dt )</td>
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</tr>
<tr>
<td>( \tan(t + s) = \frac{\tan(t) + \tan(s)}{1 - \tan(t) \tan(s)} )</td>
<td>( \tanh(t + s) = \frac{\tanh(t) + \tanh(s)}{1 + \tanh(t) \tanh(s)} )</td>
</tr>
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</table>

**Remark** (added on answer sheet). The formulas

\[
\int \frac{dv}{a^2 + vu^2} = \frac{1}{a} \tan^{-1} \frac{v}{a} + C.
\]

\[
\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u + a}{u - a} \right| + C
\]

can be related as follows. The formula

\[
\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u + a}{u - a} \right| + C
\]

can be proved with partial fractions. Assume that \(-a < u < a\) so that \((u + a)/(u - a) > 0\) and introduce the abbreviation

\[
w := \frac{1}{2a} \ln \left( \frac{u + a}{u - a} \right).
\]

Multiply by \(2a\) and exponentiate to get

\[
e^{2aw} = \frac{a + u}{a - u}
\]

---

1. This is the same table that was emailed to the class yesterday morning.
so $a e^{2aw} - u e^{2aw} = a + u$ so $a(e^{2aw} - 1) = u(e^{2aw} + 1)$ so

$$\frac{u}{a} = \frac{e^{2aw} - 1}{e^{2aw} + 1} = \frac{e^{aw} - e^{-aw}}{e^{aw} + e^{-aw}} = \tanh(aw)$$

by high school algebra. Thus $\tanh^{-1}(u/a) = aw$ so

$$w = \frac{1}{a} \tanh^{-1} \frac{u}{a}.$$ 

Now the trig functions and hyperbolic functions are related by the formulas

$$i \sin(t) = \sinh(it), \quad \cos(t) = \cosh(it), \quad i \tan(t) = \tanh(it)$$

so the substitution $u = iv$, $du = i \, dv$ gives

$$\int \frac{du}{a^2 + u^2} = \int \frac{i \, dv}{a^2 - v^2} = \frac{i}{a} \tanh^{-1} \frac{iv}{a} + C = \frac{1}{a} \tan^{-1} \frac{v}{a} + C.$$