In questions II and III use the definition you give in Problem I. Do not cite any theorem from the text, but you may use the elementary properties of inequalities without proof.

I. Define continuity. Avoid using the limit notation.

**Answer.** A function \( f : D \to \mathbb{R} \) is said to be **continuous** at a point \( x_0 \in D \) iff for every \( \varepsilon > 0 \) \( \exists \delta > 0 \) such that \( \forall x \in D \) we have

\[
|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.
\]

(\( \ast \))

**Remark.** The implication (\( \ast \)) is trivial for \( x = x_0 \) (for then \( f(x) - f(x_0) = 0 \)) so in case that \( x_0 \) is an accumulation point of the domain of \( f \) continuity at \( x_0 \) is the same as the condition that for every \( \varepsilon > 0 \) \( \exists \delta > 0 \) such that \( \forall x \in D \) we have

\[
0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon,
\]

i.e. the condition that

\[
\lim_{x \to x_0} f(x) = f(x_0).
\]

However the definition of continuity we assume that \( x_0 \in D \) but not that \( x_0 \) is an accumulation point of \( D \). Recall that if we were to drop the requirement that \( x_0 \) be an accumulation point of \( D \), then the definition of limit would be nonsense, i.e. every number \( L \) would satisfy \( \lim_{x \to x_0} f(x) = L \).

II. State and prove a theorem which says that the product of two continuous functions is continuous.

**Answer. Theorem.** Assume that \( f, g : D \to \mathbb{R} \) are both continuous at \( x_0 \in D \) and that \( h : D \to \mathbb{R} \) is defined by \( h(x) = f(x)g(x) \) for \( x \in D \). Then \( h \) is continuous at \( x_0 \).

**Proof.** Choose \( \varepsilon > 0 \). Let

\[
\varepsilon' = \min \left( 1, \frac{\varepsilon}{|f(x_0)| + 1 + |g(x_0)|} \right).
\]

As \( f \) is continuous at \( x_0 \) there exists \( \delta_1 > 0 \) such that for all \( x \in D \) we have

\[
|x - x_0| < \delta_1 \implies |f(x) - f(x_0)| < \varepsilon'.
\]

As \( g \) is continuous at \( x_0 \) there exists \( \delta_2 > 0 \) such that for all \( x \in D \) we have

\[
|x - x_0| < \delta_2 \implies |g(x) - g(x_0)| < \varepsilon'.
\]
Let \( \delta = \min(\delta_1, \delta_2) \). Choose \( x \in D \) such that \( |x - x_0| < \delta \). Then \( |f(x)| \leq |f(x_0)| + 1 \) (as \( \delta \leq \delta_1 \) and \( \varepsilon' \leq 1 \)) so

\[
|f(x)| |g(x) - g(x_0)| \leq (|f(x_0)| + 1)|g(x) - g(x_0)| < (|f(x_0)| + 1)\varepsilon'.
\]

Also

\[
|f(x) - f(x_0)| |g(x_0)| < \varepsilon' |g(x_0)|.
\]

Hence

\[
|h(x) - h(x_0)| = |f(x)g(x) - f(x_0)g(x_0)|
\]

\[
\leq |f(x)| |g(x) - g(x_0)| + |f(x) - f(x_0)| |g(x_0)|
\]

\[
< (|f(x_0)| + 1)\varepsilon' + \varepsilon' |g(x_0)| = \varepsilon
\]

as required.

\[ \square \]

**III.** State and prove a theorem which says that the composition of two continuous functions is continuous.

**Answer.** Theorem. Let \( g : D \to E, f : E \to \mathbb{R}, \) and \( x_0 \in D \) so \( y_0 := g(x_0) \in E \). Assume \( g \) is continuous at \( x_0 \) and \( f \) is continuous at \( y_0 \). Then \( f \circ g \) is continuous at \( x_0 \).

**Proof.** Choose \( \varepsilon > 0 \). As \( f \) is continuous at \( y_0 \) there exists \( \eta > 0 \) such that for all \( y \in E \) we have

\[
|y - y_0| < \eta \implies |f(y) - f(y_0)| < \varepsilon.
\]

(1)

As \( g \) is continuous at \( x_0 \) there exists \( \delta > 0 \) such that for all \( x \in D \) we have

\[
|x - x_0| < \delta \implies |g(x) - g(x_0)| < \eta.
\]

(2)

Choose \( x \in D \) such that \( |x - x_0| < \delta \). Then \( g(x) \in E \) (as \( g : D \to E \)) and \( |g(x) - g(x_0)| < \eta \) by (2). Hence, taking \( y = g(x) \) in (1), gives

\[
|f \circ g(x) - f \circ g(x_0)| = |f(g(x)) - f(g(x_0))| < \varepsilon
\]

as required.

\[ \square \]

**IV.** Assume that \( F : \mathbb{R} \setminus \{b\} \to \mathbb{R} \), that \( \lim_{y \to b} F(y) = L \), that \( g : \mathbb{R} \to \mathbb{R} \) is continuous and one-one, and \( g(a) = b \). Show that \( \lim_{x \to a} F(g(x)) = L \).

**Answer.** Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(y) = F(y) \) for \( y \neq b \) and \( f(b) = L \). Then \( f \) is continuous at \( b \), so by problem III \( g \circ f \) is continuous at \( a \). Hence \( \lim_{x \to a} f(g(x)) = f(g(a)) = f(b) = L \). But since \( g \) is one-one we have that \( g(x) \neq b \) for \( x \neq a \) so \( f(g(x)) = F(g(x)) \) for \( x \neq a \) and hence

\[
\lim_{x \to a} F(g(x)) = f(g(a)) = \lim_{x \to a} f(g(x)) = L
\]
as required.

**V.** Prove that \( \lim_{x \to \pi/2} \frac{\cos x}{\frac{\pi}{2} - x} = 1 \). You may use the following facts from calculus without proof:

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \cos(x) = \sin(\frac{\pi}{2} - x).
\]

**Answer.** This is an immediate consequence of Problem IV with \( g(x) = \frac{\pi}{2} - x, a = 0, b = g(a) = \frac{\pi}{2}, \)

\[
F(y) = \frac{\cos y}{\frac{\pi}{2} - y}, \quad \text{so} \quad F(g(x)) = \frac{\sin x}{x}.
\]

**VI.** What is the domain of the composition \( F \circ g \) in Problem IV? In Problem IV, why did I assume that \( g \) is one-one? What is the relation between Problem V and Problem IV? Problem V and Problem III?

**Answer.** The domain of the composition \( F \circ g \) is the set

\[
\text{Dom}(F \circ g) = \{ x \in \text{Dom}(g) : g(x) \in \text{Dom}(F) \}.
\]

Since \( \text{Dom}(g) = \mathbb{R} \), \( \text{Dom}(F) = \mathbb{R} \setminus \{b\} \), \( g(a) = b \), and \( g \) is one-one, we get \( \text{Dom}(F \circ g) = \mathbb{R} \setminus \{a\} \). We assumed that \( g \) is one-one to insure that \( a \) is a limit point of \( F \circ g \). As the above proofs show we can prove IV from III and V from IV.