1.4 Theorem A Cauchy sequence is bounded.

Exercise 44. Let $S \subset \mathbb{R}$ be a nonempty set of real numbers which is bounded above and $s = \sup S$. Then for all $\varepsilon > 0$ there exists $x \in S$ such that $s - \varepsilon < x \leq s$.

Exercise 45. Let $S \subset \mathbb{R}$ be a nonempty set of real numbers which is bounded below and $\ell = \inf S$. Then for all $\varepsilon > 0$ there exists $x \in S$ such that $\ell \leq x < \ell + \varepsilon$.

1.7 Theorem. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then there exists a real number $A$ such that $\{x_n\}_{n=1}^{\infty}$ converges to $A$.

Proof. By Theorem 1.4 there exist numbers $L$ and $U$ with

$$L \leq x_n \leq U \tag{1}$$

for all $n \in \mathbb{J}$. By (1) the set $\{x_n : k \leq n\}$ is bounded above so we may define

$$s_k := \sup\{x_n : k \leq n\}. \tag{2}$$

Then by (1) and (2)

$$L \leq x_k \leq s_k \tag{3}$$

for all $k$. We have

$$p \leq q \implies s_q \leq s_p \tag{4}$$

as $\{x_n : q \leq n\} \subset \{x_n : p \leq n\}$ for $p \leq q$. By (3) the set $\{s_k : k \in \mathbb{J}\}$ is bounded below so we may define

$$A := \inf\{s_k : k \in \mathbb{J}\}. \tag{5}$$

Choose $\varepsilon > 0$. We must find $N$ so that

$$A - \varepsilon < x_n < A + \varepsilon \tag{*}$$

for $n \geq N$. Because the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy there exists $M$ with

$$-\frac{\varepsilon}{2} < x_n - x_m < \frac{\varepsilon}{2} \tag{6}$$

for $n, m \geq M$. By (5) and Exercise 45 there exists $k$ with

$$A \leq s_k < A + \frac{\varepsilon}{2}. \tag{7}$$

Let $N = \max(k, M)$. Then by (4) and (7) we have

$$A \leq s_N < A + \frac{\varepsilon}{2} \tag{8}$$

By (2) and Exercise 44 there exists $m \geq N$ such that

$$s_N - \frac{\varepsilon}{2} < x_m \leq s_N. \tag{9}$$

By (8) and (9)

$$A - \frac{\varepsilon}{2} < x_m < A + \frac{\varepsilon}{2}. \tag{10}$$

Adding (6) and (10) gives (*) for $n \geq N$ as required. □