Math 421 – Quiz VIII– Feb 20, 2004– Answers

Prove

**The Sandwich Theorem.** Assume that \(\{a_n\}_{n=1}^{\infty}\) and \(\{b_n\}_{n=1}^{\infty}\) are sequences which converge to \(A\) and that

\[
a_n \leq c_n \leq b_n \quad (\ast)
\]

for all \(n\). Then \(\{c_n\}_{n=1}^{\infty}\) converges to \(A\).

**Proof.** By hypothesis

\[
\forall \varepsilon > 0 \ \exists N_1 \in J \text{ such that } |a_n - A| < \varepsilon \text{ for } n \geq N_1, \quad (1)
\]

and

\[
\forall \varepsilon > 0 \ \exists N_2 \in J \text{ such that } |b_n - A| < \varepsilon \text{ for } n \geq N_2. \quad (2)
\]

Choose \(\varepsilon > 0\) and let \(N_1\) be as in (1) and \(N_2\) be as in (2). Then by (1) we have \(A - \varepsilon < a_n\) for \(n \geq N_1\) and by (2) we have \(b_n < A + \varepsilon\) for \(n \geq N_2\). Combining with (\(\ast\)) gives

\[
A - \varepsilon < a_n \leq c_n \leq b_n < A + \varepsilon
\]

for \(n \geq N : = \max(N_1, N_2)\). This implies that \(|A - c_n| < \varepsilon\) for \(n \geq N\) as required. \(\square\)