Continuity and Uniform Continuity

521

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1. Throughout S will denote a subset of the real numbers $\mathbf{R}$ and $f : S \to \mathbf{R}$ will be a real valued function defined on S. The set S may be bounded like

$$S = (0, 5) = \{x \in \mathbf{R} : 0 < x < 5\}$$

or infinite like

$$S = (0, \infty) = \{x \in \mathbf{R} : 0 < x\}.$$

It may even be all of $\mathbf{R}$. The value $f(x)$ of the function f at the point $x \in S$ will be defined by a formula (or formulas).

Definition 2. The function f is said to be **continuous on** S iff

$$\forall x_0 \in S\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in S \Big[|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \Big].$$

Hence f is not continuous on S iff

$$\exists x_0 \in S\ \exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in S \Big[|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \geq \varepsilon \Big].$$

Definition 3. The function f is said to be **uniformly continuous on** S iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_0 \in S\ \forall x \in S \Big[|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \Big].$$

Hence f is not uniformly continuous on S iff

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x_0 \in S\ \exists x \in S \Big[|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \geq \varepsilon \Big].$$

4. The only difference between the two definitions is the order of the quantifiers. When you prove f is continuous your proof will have the form

> *Choose* $x_0 \in S$. *Choose* $\varepsilon > 0$. *Let* $\delta = \delta(x_0, \varepsilon)$. *Choose* $x \in S$. *Assume* $|x - x_0| < \delta$. $\cdots$ *Therefore* $|f(x) - f(x_0)| < \varepsilon$.

The expression for $\delta(x_0, \varepsilon)$ can involve both x_0 and ε but must be independent of x. The order of the quanifiers in the definition signals this; in the proof x has not yet been chosen at the point where δ is defined so the definition of δ must not involve x. (The $\cdots$ represent the proof that $|f(x) - f(x_0)| < \varepsilon$ follows from the earlier steps in the proof.) When you prove f is uniformly continuous your proof will have the form

> *Choose* $\varepsilon > 0$. *Let* $\delta = \delta(\varepsilon)$. *Choose* $x_0 \in S$. *Choose* $x \in S$. *Assume* $|x - x_0| < \delta$. $\cdots$ *Therefore* $|f(x) - f(x_0)| < \varepsilon$.

so the expression for δ can only involve ε and must not involve either x or x_0.

It is obvious that a uniformly continuous function is continuous: if we can find a δ which works for all x_0, we can find one (the same one) which works for any particular x_0. We will see below that there are continuous functions which are not uniformly continuous.

Example 5. Let $S = \mathbf{R}$ and $f(x) = 3x + 7$. Then f is uniformly continuous on S.

Proof. Choose $\varepsilon > 0$. Let $\delta = \varepsilon/3$. Choose $x_0 \in \mathbf{R}$. Choose $x \in \mathbf{R}$. Assume $|x - x_0| < \delta$. Then

$$|f(x) - f(x_0)| = |(3x + 7) - (3x_0 + 7)| = 3|x - x_0| < 3\delta = \varepsilon.$$

□

Example 6. Let $S = \{x \in \mathbf{R} : 0 < x < 4\}$ and $f(x) = x^2$. Then f is uniformly continuous on S.

Proof. Choose $\varepsilon > 0$. Let $\delta = \varepsilon/8$. Choose $x_0 \in S$. Choose $x \in S$. Thus $0 < x_0 < 4$ and $0 < x < 4$ so $0 < x + x_0 < 8$. Assume$|x - x_0| < \delta$. Then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = (x + x_0)|x - x_0| < (4 + 4)\delta = \varepsilon.$$

□

7. In both of the preceeding proofs the function f satisfied an inequality of form

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \tag{1}$$

for $x_1, x_2 \in S$. In Example 5 we had

$$|(3x_1 + 7) - (3x_2 + 7)| \leq 3|x_1 - x_2|$$

and in Example 6 we had

$$|x_1^2 - x_2^2| \leq 8|x_1 - x_2|$$

for $0 < x_1, x_2 < 4$. An inequality of form (1) is called a **Lipschitz inequality** and the constant M is called the corresponding **Lipschitz constant**.

Theorem 8. If f satisfies (1) for $x_1, x_2 \in S$, then f is uniformly continuous on S.

Proof. Choose $\varepsilon > 0$. Let $\delta = \varepsilon/M$. Choose $x_0 \in S$. Choose $x \in S$. Assume that $|x - x_0| < \delta$. Then

$$|f(x) - f(x_0)| \leq M|x - x_0| < M\delta = \varepsilon.$$

□

9. The Lipschitz constant depend might depend on the interval. For example,

$$|x_1^2 - x_2^2| = (x_1 + x_2)|x_1 - x_2| \leq 2a|x_1 - x_2|$$

for $0 < x_1, x_2 < a$ but the function $f(x) = x^2$ does not satisfy a Lipschitz inequality on the whole interval $(0, \infty)$ since

$$|x_1^2 - x_2^2| = (x_1 + x_2)|x_1 - x_2| > M|x_1 - x_2|$$

if $x_1 = M$ and $x_2 = x_1 + 1$. In fact,

Example 10. The function $f(x) = x^2$ is continuous but not uniformly continuous on the interval $S = (0, \infty)$.

Proof. We show f is continuous on S, i.e.

$$\forall x_0 \in S \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in S \left[|x - x_0| < \delta \implies |x^2 - x_0^2| < \varepsilon \right].$$

Choose x_0. Let $a = x_0 + 1$ and $\delta = \min(1, \varepsilon/2a)$. (Note that δ depends on x_0 since a does.) Choose $x \in S$. Assume $|x - x_0| < \delta$. Then $|x - x_0| < 1$ so $x < x_0 + 1 = a$ so $x, x_0 < a$ so

$$|x^2 - x_0^2| = (x + x_0)|x - x_0| \le 2a|x - x_0| < 2a\delta \le 2a\frac{\varepsilon}{2a} = \varepsilon$$

as required.

We show that f is not uniformly continuous on S, i.e.

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x_0 \in S\ \exists x \in S \Big[|x - x_0| < \delta \text{ and } |x^2 - x_0^2| \ge \varepsilon \Big].$$

Let $\varepsilon = 1$. Choose $\delta > 0$. Let $x_0 = 1/\delta$ and $x = x_0 + \delta/2$. Then $|x - x_0| = \delta/2 < \delta$ but

$$|x^2 - x_0^2| = \left| \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2 \right| = 1 + \frac{\delta^2}{4} > 1 = \varepsilon$$

as required. (Note that x_0 is large when δ is small.) □

11. According to the Mean Value Theorem from calculus for a differentiable function f we have

$$f(x_1) - f(x_2) = f'(c)(x_2 - x_1).$$

for *some* c between x_1 and x_2. (The slope $(f(x_1) - f(x_2))/(x_1 - x_2)$ of the secant line joining the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph is the same as the slope $f'(c)$ of the tangent point at the intermediate point $(c, f(c))$.) If x_1 and x_2 lie in some interval S and $|f'(c)| \le M$ for *all* $c \in S$ we conclude that the Lipschitz inequality (1) holds on S. We don't want to use the Mean Value Theorem without first proving it, but we certainly can use it to guess an appropriate value of M and then prove the inequality by other means.

12. For example, consider the function $f(x) = x^{-1}$ defined on the interval $S = (a, \infty)$ where $a > 0$. For $x_1, x_2 \in S$ the Mean Value Theorem says that $x_1^{-1} - x_2^{-1} = -c^{-2}(x_1 - x_2)$ where c is between x_1 and x_2. If $x_1, x_2 \in S$ then $c \in S$ (as c is between x_1 and x_2) and hence $c > a$ so $c^{-2} < a^{-2}$. We can prove the inequality

$$|x_1^{-1} - x_2^{-1}| \le a^{-2}|x_2 - x_2|$$

for $x_1, x_2 \geq a$ as follows. First $a^2 \leq x_1 x_2$ since $a \leq x_1$ and $a \leq x_2$. Then

$$|x_1^{-1} - x_2^{-1}| = \frac{|x_1 - x_2|}{x_1 x_2} \leq \frac{|x_1 - x_2|}{a^2} \tag{2}$$

where we have used the fact that $\alpha^{-1} < \beta^{-1}$ if $0 < \alpha < \beta$. It follows that that the function $f(x)$ is uniformly continuous on any interval (a, ∞) where $a > 0$. Notice however that the Lipschitz constant $M = a^{-2}$ depends on the interval. In fact, the function $f(x) = x^{-1}$ does *not* satisfy a Lipshitz inequality on the interval $(0, \infty)$.

13. We can discover a Lipscitz inequality for the square root function $f(x) = \sqrt{x}$ in much the same way. Consider the function $f(x) = \sqrt{x}$ defined on the interval $S = (a, \infty)$ where $a > 0$. For $x_1, x_2 \in S$ the Mean Value Theorem says that $\sqrt{x_1} - \sqrt{x_2} = (x_1 - x_2)/(2\sqrt{c})$ where c is between x_1 and x_2. If $x_1, x_2 \in S$ then $c \in S$ (as c is between x_1 and x_2) and hence $c > a$ so $(2\sqrt{c})^{-1} < (2\sqrt{a})^{-1}$. We can prove the inequality

$$|\sqrt{x_1} - \sqrt{x_2}| \leq \frac{|x_1 - x_2|}{2\sqrt{a}} \tag{3}$$

for $x_1, x_2 \geq a$ as follows: Divide the equation

$$(\sqrt{x_1} - \sqrt{x_2})(\sqrt{x_1} + \sqrt{x_2}) = ((\sqrt{x_1})^2 - (\sqrt{x_2})^2) = x_1 - x_2$$

by $(\sqrt{x_1} + \sqrt{x_2})$, take absolute values, and use $(\sqrt{x_1} + \sqrt{x_2}) \geq 2\sqrt{a}$. Again the Lipschitz constant $M = (2\sqrt{a})^{-1}$ depends on the interval and the function does *not* satisfy a Lipschitz inequality on the interval $(0, \infty)$.

Example 14. The function $f(x) = x^{-1}$ is continuous but not uniformly continuous on the interval $S = (0, \infty)$.

Proof. We show f is continuous on S, i.e.

$$\forall x_0 \in S \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in S \left[|x - x_0| < \delta \implies \left| \frac{1}{x} - \frac{1}{x_0} \right| < \varepsilon \right].$$

Choose x_0. Let $a = x_0/2$ and $\delta = \min(x_0 - a, a^2\varepsilon)$. Choose $x \in S$. Assume $|x - x_0| < \delta$. Then $x_0 - x \leq |x - x_0| < x_0 - a$ so $-x < -a$ so $a < x$ so $x, x_0 < a$ so by (2)

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| \leq \frac{|x_1 - x_2|}{a^2} < \frac{\delta}{a^2} \leq \frac{a^2\varepsilon}{a^2} = \varepsilon$$

as required.

We show that f is not uniformly continuous on S, i.e.

$$\exists\varepsilon > 0 \; \forall\delta > 0 \; \exists x_0 \in S \; \exists x \in S \left[|x - x_0| < \delta \text{ and } \left| \frac{1}{x} - \frac{1}{x_0} \right| \geq \varepsilon \right].$$

Let $\varepsilon = 1$. Choose $\delta > 0$. Let $x_0 = \min(\delta, 1)$ and $x = x_0/2$. Then $|x - x_0| = x_0/2 \leq \delta/2 < \delta$ but

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{1}{x_0/2} - \frac{1}{x_0} \right| = \frac{1}{x_0} \geq 1 = \varepsilon$$

as required. □

Example 15. The function $f(x) = \sqrt{x}$ is uniformly continuous on the set $S = (0, \infty)$.

Remark 16. This example shows that a function can be uniformly continuous on a set even though it does not satisfy a Lipschitz inequality on that set, i.e. the method of Theorem 8 is not the only method for proving a function uniformly continuous. The proof we give will use the following idea. After choosing $\varepsilon > 0$ we specify two numbers a and b which will depend on ε. These numbers will satisfy $0 < a < b$. We will choose δ so that (among other things) $\delta < b - a$. Then after we choose $x, x_0 \in S$ and assume that $|x - x_0| < \delta$ we will be able to conclude that either both x_0 and x are less than b or both are greater than a. We will choose b so small that $\sqrt{x}$ and $\sqrt{x_0}$ are within ε of zero for $x, x_0 < b$. We will use a Lipschitz inequality to handle the case where $x, x_0 > a$. We give the details of this proof after some preliminary lemmas. The only properties of that square root function that we will use are that $\sqrt{x}$ is defined for $x \geq 0$ and satisfies

$$\sqrt{x} \geq 0, \qquad (\sqrt{x})^2 = x, \qquad \sqrt{x^2} = x.$$

Lemma 17. *The square root function is increasing:*

$$0 \leq a < b \implies \sqrt{a} < \sqrt{b}.$$

Proof. Assume $0 \leq a < b$. If $\sqrt{b} \leq \sqrt{a}$ then $b = (\sqrt{b})^2 \leq (\sqrt{a})^2 = a$ contradicting $a < b$. Hence $\sqrt{a} < \sqrt{b}$. □

Lemma 18. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ *for* $a, b \geq 0$.

Proof. $(\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$ so $\sqrt{a}\sqrt{b} = \sqrt{(\sqrt{a}\sqrt{b})^2} = \sqrt{ab}$. □

Lemma 19. *Assume $a < b$. Then for any two numbers x and y at least one of the four alternatives*

$$\begin{array}{llll} (i) & x < b \ \& \ y < b, & (ii) & a \leq x \ \& \ a \leq y, \\ (iii) & x < a \ \& \ b \leq y, & (iv) & y < a \ \& \ b \leq x. \end{array}$$

Proof. Exactly one of the three alternatives $x < a$, $a \leq x < b$, $b \leq x$ holds and exactly one of the three alternatives $y < a$, $a \leq y < b$, $b \leq y$ holds. There are thus nine cases which we can arrange in a table:

	$x < a$	$a \leq x < b$	$b \leq x$
$y < a$	(i)	(i)	(iv)
$a \leq y < b$	(i)	$(i), (ii)$	(ii)
$b \leq y$	(iii)	(ii)	(ii)

In each entry of the table we have indicated the alternative $(i - iv)$ which holds in the corresponding case. □

Proof. Now we prove what is claimed in Example 15, viz. that the square root function is uniformly continuous on the positive real numbers, i.e.

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, x_0 > 0 \Big[|x - x_0| < \delta \implies |\sqrt{x} - \sqrt{x_0}| < \varepsilon \Big].$$

Choose $\varepsilon > 0$. Let $\delta = \min(\varepsilon^2/2, \sqrt{2}\varepsilon^2)$. Choose $x, x_0 > 0$. Assume $|x - x_0| < \delta$. Read $a = \varepsilon^2/2$ and $b = \varepsilon^2$ in Lemma 19: we need consider only four cases:

$$\begin{array}{llll} \text{(i)} & x < \varepsilon^2 \ \& \ x_0 < \varepsilon^2, & \text{(ii)} & \varepsilon^2/2 \leq x \ \& \ \varepsilon^2/2 \leq x_0, \\ \text{(iii)} & x < \varepsilon^2/2 \ \& \ \varepsilon^2 \leq x_0, & \text{(iv)} & x_0 < \varepsilon^2/2 \ \& \ \varepsilon^2 \leq x. \end{array}$$

Cases (iii) and (iv) contradict the assumption that $|x - x_0| < \delta \leq \varepsilon^2/2 = b - a$ so we need only consider cases (i) and (ii). In case (i) we have $\sqrt{x} < \varepsilon$ and $\sqrt{x_0} < \varepsilon$ by Lemma 17 so $|\sqrt{x} - \sqrt{x_0}| \leq \max(\sqrt{x}, \sqrt{x_0}) < \varepsilon$. In case (ii) we use the inequality (3) and get

$$|\sqrt{x} - \sqrt{x_0}| \leq \frac{|x - x_0|}{2\sqrt{\varepsilon^2/2}} = \frac{|x - x_0|}{\sqrt{2}\varepsilon} < \frac{\delta}{\sqrt{2}\varepsilon} \leq \varepsilon.$$

□

Remark 20. Of course, $\min(\varepsilon^2/2, \sqrt{2}\varepsilon^2) = \varepsilon^2/2$. The more complicated formula is given to help the reader understand how the proof was discovered.

Remark 21. We can also prove that the square root function is uniformly continuous by taking $\delta = \varepsilon^2$ and using the inequality

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b}.$$

(To prove the inequality square both sides.) If $|x - x_0| < \varepsilon^2$ then either $x_0 \le x < x_0+\varepsilon^2$ (which implies $\sqrt{x_0} \le \sqrt{x} < \sqrt{x_0}+\varepsilon$) or else $x \le x_0 < x+\varepsilon^2$ (which implies $\sqrt{x} \le \sqrt{x_0} < \sqrt{x}+\varepsilon$). In either case $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$. This proof is simpler than the one given but is harder to find since it uses a clever inequality which the reader might not think of.

Example 22. The function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

is not continuous.

Proof. In other words,

$$\exists x_0 \in \mathbf{R}\ \exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbf{R} \Big[|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \ge \varepsilon \Big].$$

Let $x_0 = 0$. (No other value of x_0 will work here.) Let $\varepsilon = 1$. Choose $\delta > 0$. Let $x = \delta/2$. Then $|x - x_0| = \delta/2 < \delta$ but $|f(x) - f(x_0)| = |1 - 0| = 1 \ge \varepsilon$. $\square$