

Pep Talk

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The main prerequisites are an undergraduate course in point set topology (metric spaces, topological spaces, open and closed sets, compactness, homeomorphisms) and an undergraduate course in abstract algebra (groups, rings, fields, and vector spaces). Try to prove the following to get the idea of the subject.

1. The five classical regular polyhedra are the tetrahedron ($V = 4$, $E = 6$, $F = 4$), the cube ($V = 8$, $E = 12$, $F = 6$), the octahedron ($V = 6$, $E = 12$, $F = 8$), the dodecahedron ($V = 20$, $E = 30$, $F = 12$), the icosahedron ($V = 12$, $E = 30$, $F = 20$). In all five cases the **Euler characteristic**

$$\chi := V - E + F$$

is two. (Here V , E , and F are the number of vertices, edges, and faces respectively.) It is easy to see (radial projection) that each of these polyhedra is homeomorphic to

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The Euler characteristic is a topological invariant: if a polyhedron is homeomorphic to the sphere, then $V - E + F = 2$.

2. The n -dimensional unit **disk** is the set

$$\mathbb{D}^n := \{x \in \mathbb{R}^n : |x| \leq 1\}.$$

The **Brouwer fixed point theorem** says that any map $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$ has a fixed point, i.e. there exists $x \in \mathbb{D}^n$ with $f(x) = x$. This is easy to prove when $n = 1$.

3. A vector field on a manifold $M \subset \mathbb{R}^n$ is a map $v : M \rightarrow \mathbb{R}^n$ such that $v(x)$ is tangent to M at x for each $x \in M$. For example, a vector field on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a map $v : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ such that $\langle v(x), x \rangle = 0$. It is easy to construct a nowhere vanishing vector field on an odd dimensional sphere $\mathbb{S}^{2m-1} \subset \mathbb{R}^{2m} = \mathbb{C}^m$ (e.g. $v(x) = ix$), but every vector field on an even dimensional sphere must have at least one zero. This theorem is sometimes called the **Hairy Ball Theorem**.

4. In set theory a function $f : X \rightarrow Y$ between two sets is injective if and only if it has a left inverse, and surjective if and only if it has a right inverse. (The latter is the **Axiom of Choice**: if f is surjective, then $f^{-1}(y) \neq \emptyset$ for $y \in Y$ and there is a function g with $g(y) \in f^{-1}(y)$.) In linear algebra this principle holds as well: if $A : V \rightarrow W$ is a linear map between two vector spaces then A is injective if and only if there is a linear map $B : W \rightarrow V$ such that $BA = \text{id}_V$ and A is surjective if and only if there is a linear map $B : W \rightarrow V$ such that $AB = \text{id}_W$. This principle fails in topology. For example, the Hairy Ball Theorem says that the surjective map

$$\{(x, v) \in \mathbb{S}^2 \times \mathbb{R}^3 : \langle x, v \rangle = 0, v \neq 0\} \rightarrow \mathbb{S}^2 : (x, v) \mapsto x$$

has no (continuous) left inverse. Similarly, the inclusion $\mathbb{S}^n \rightarrow \mathbb{D}^n$ is continuous and injective but we will prove that it has no left inverse. In topology a right inverse to a map $p : E \rightarrow B$ is called a **section**, particularly in contexts where the map p is called a **projection**. A left inverse to a map $\iota : A \rightarrow X$ is called a **retraction**, particularly in contexts where the map ι is the inclusion of a subset A into its ambient space X . Thus the fact that there is no left inverse to the inclusion $\mathbb{S}^n \rightarrow \mathbb{D}^{n+1}$ is called the **No Retraction Theorem**. The case $n = 0$ of the No Retraction Theorem is easy.

Remark 5. The No Retraction Theorem easily implies the Brouwer Fixed Point Theorem as follows. If $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$ has no fixed point, each point $x \in \mathbb{D}^n$ determines a line $L_x := \{x + t(f(x) - x) : t \in \mathbb{R}\}$ which passes through both x and $f(x)$. This line intersects the sphere $\|x\|^2 = 1$ at those values of t for which

$$1 = \|x + t(f(x) - x)\|^2 = t^2\|x - f(x)\|^2 + 2t\langle f(x) - x, x \rangle + \|x\|^2.$$

When $\|x\|, \|f(x)\| \leq 1$ the unique non positive solution t is

$$\tau(x) = \frac{-\langle f(x) - x, x \rangle - \sqrt{\langle f(x) - x, x \rangle^2 + \|x - f(x)\|^2(1 - \|x\|^2)}}{\|x - f(x)\|^2}$$

and $r : \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$ defined by $r(x) = x + \tau(x)(f(x) - x)$ is a retraction. To prove that $r(x) = x$ when $\|x\| = 1$ we must show that $\|x\|^2 = 1$ implies that $\tau(x) = 0$, i.e. that $\langle f(x) - x, x \rangle \leq 0$, i.e. that $\langle f(x), x \rangle \leq \langle x, x \rangle = \|x\|^2 = 1$. This follows from the Schwartz inequality as $\|f(x)\| \leq 1$.

6. The **Borsuk Ulam Theorem** says that for any map $f : \mathbb{S}^n \rightarrow \mathbb{R}^n$ there is a point $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. When $n = 2$ this implies that there is always a pair of antipodal points on earth where the temperature and barometric pressure are the same. Even the case $n = 1$ requires a bit of thought.