1 Preliminaries

1.1. Unless otherwise specified, the word \textit{space} means \textit{topological space} and the word \textit{map} means \textit{continuous function} between two topological spaces. The notation \( f : (X, A) \to (Y, B) \) means \( f : X \to Y, \ A \subset X, \ B \subset Y, \) and \( f(A) \subset B. \) It is often convenient to fix a \textbf{base point} \( x_0 \in X. \) The pair \( (X, x_0) \) is then called a \textbf{pointed space} and the notation \( f : (X, x_0) \to (Y, y_0) \) means \( f : X \to Y, \ x_0 \in X, \ y_0 \in Y, \) and \( f(x_0) = y_0. \)

1.2. The \textbf{disjoint union} of an indexed collection \( \{Y_\alpha\}_{\alpha \in \Lambda} \) is by definition the set \( \bigsqcup_{\alpha \in \Lambda} Y_\alpha := \{(y, \alpha) : y \in Y_\alpha\}. \) There is a projection \( \bigsqcup_{\alpha \in \Lambda} Y_\alpha \to \bigcup_{\alpha \in \Lambda} Y_\alpha : (y, \alpha) \mapsto y \) which is a bijection if and only if the sets \( Y_\alpha \) are pairwise disjoint.

1.3. By a \textbf{disk} we understand a space homeomorphic to the \textbf{standard closed unit disk} \( \mathbb{D}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}. \) We may on occasion use other models for the disk, e.g. the \textbf{unit cube} \( \mathbb{I}^n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1\} \) or the \textbf{standard simplex} \( \Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{I}^{n+1} : x_0 + x_1 + \cdots + x_n = 1\}. \) The spaces \( \mathbb{D}^n, \mathbb{I}^n, \) and \( \Delta^n \) are homeomorphic; homeomorphisms can be constructed using radial projection. We use the terms \( n \)-disk, \( n \)-cube, or \( n \)-simplex to indicate the dimension. When \( D \) is an \( n \)-disk we denote by \( \partial D \) the image of \( \mathbb{S}^{n-1} := \partial \mathbb{D}^n := \{x \in \mathbb{R}^n : \|x\| = 1\} \) under a homeomorphism \( \mathbb{D}^n \to D. \) By the Brouwer Invariance of Domain Theorem the subset \( \partial D \subset D \) is independent of the choice of the homeomorphism used to define it but for the definitions in this section the reader may take the
term disk to signify one of $D^n$, $I^n$, or $\Delta^n$ and define $\partial D$ in each of these cases separately. Thus

$$\partial I^n := \bigcup_{i=1}^{n} I^{i-1} \times \{0,1\} \times I^{n-i}$$

and

$$\partial \Delta^n := \bigcup_{k=0}^{n} \iota_k(\Delta^{n-1})$$

where the inclusion $\iota_k : \Delta^{n-1} \to \Delta^n$ is defined by $\iota_k(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, x_{k+1}, \ldots, x_n)$.

The set $\partial D$ is called the boundary of the disk but the reader is cautioned that in point set topology this term depends on the ambient space (in this case $\mathbb{R}^n$). Thus (In point set topology the boundary of a set is the intersection of its closure with the closure of its complement so the set theoretic boundary of $\Delta^n$ as a subset of $\mathbb{R}^{n+1}$ is $\Delta^n$ itself.) We may call a disk $D$ a closed disk to distinguish it from the open disk $D \setminus \partial D$.

1.4. A path in $X$ is a map $\gamma : I \to X$ and a loop in $X$ is a path with $\gamma(0) = \gamma(1)$. The words are also used more generally, for example a map $\gamma : [a, b] \to X$ might also be called a path and a map $\gamma : \mathbb{S}^1 \to X$ might also be called a loop. A space $X$ is called path connected iff for all $x, y \in X$ there is a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A space is said to have a property locally iff every point has arbitrarily small neighborhoods which have the property. For example, a space $X$ is locally path connected iff for every $x \in X$ and every neighborhood $V$ of $x$ in $X$ there is a path connected neighborhood $U$ of $x$ with $U \subset V$.

Exercise 1.5. For any space $X$ define an equivalence relation by $x \equiv y$ iff there is a path $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. The equivalence classes are called the path components of $X$. Show that the following are equivalent.

(i) The space $X$ is locally path connected, i.e. for every $x_0 \in X$ and every neighborhood $V$ of $x_0$ there is an open set $U$ with $x_0 \in U \subset V$ and such that any two points of $U$ can be joined by a path in $U$.

(ii) For every $x \in X$ and every neighborhood $V$ of $x$ in $X$ there is a a neighborhood $U$ of $x$ with $U \subset V$ and such that any two points of $U$ can be joined by a path in $V$.

(iii) The path components of every open set are open.

Solution. That (i) $\implies$ (ii) is immediate: a path in $U$ is a fortiori a path in $V$. Assume (ii). Choose an open set $V$ and a path component $C$ of $V$. (We will show $C$ is open.) Choose $x_0 \in C$. By (ii) there is an open set $U$ containing $x_0$ such that any two points of $U$ lie in the same path component of $V$. In particular any point of $U$ lies in the same path component of $C$ as $x_0$, i.e. $U \subset C$. Thus $C$ is open. This proves (iii). Finally assume (iii). Choose $x_0 \in X$ and a neighborhood $V$ of $x_0$. Let $U$ be the path component of $V$ containing $x_0$. By (iii), $U$ is open. This proves (i).
Remark 1.6. The whole space $X$ is an open set so it follows from (iii) that if $X$ is locally path connected, then the path components are open.

Remark 1.7. It is a theorem of point set topology that a compact, connected, and locally connected metric space is path connected. (See [4] page 116.) However, a compact metric space can be quite nasty. Consider for example the exercises in page 18 Chapter 0 of [3]. Pathological examples are important for understanding proofs, but for the most we will concentrate on nice spaces (finite cell complexes – See Definition 5.2 below).

1.8. A homotopy is a family of maps $\{f_t : X \to Y\}_{t \in I}$ such that the evaluation map

$$X \times I \to Y : (x, t) \mapsto f_t(x)$$

is continuous. We do not distinguish between the homotopy and the corresponding evaluation map. We say two maps $f, g : X \to Y$ are homotopic and write $f \simeq g$ iff there is a homotopy with $f_0 = f$ and $f_1 = g$. When we say that $f, g : (X, A) \to (Y, B)$ are homotopic it is understood that each stage $f_t$ of the homotopy is also a map of pairs, i.e. $f_t(A) \subset B$. If $A \subset X$ we say $f$ and $g$ are homotopic relative to $A$ and write $f \simeq g$ (rel $A$) iff there is a homotopy $\{f_t\}_{t \in I}$ between $f$ and $g$ with $f_t(a) = f_0(a)$ for all $a \in A$ and $t \in I$.

Remark 1.9. When $X$ is compact and the space $C(X, Y)$ of all continuous maps from $X$ to $Y$ is endowed with the compact open topology, two maps $f, g \in C(X, Y)$ are homotopic if and only if they belong to the same path component of $C(X, Y)$. Thus (in this situation at least) a homotopy is truly a path of maps.

Definition 1.10. We say spaces $X$ and $Y$ are homotopy equivalent or have the same homotopy type and write $X \simeq Y$ iff there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.

Definition 1.11. A space $X$ is called contractible iff it is homotopy equivalent to a point, i.e. iff there is a homotopy $f_t : X \to X$ with $f_0 = \text{id}_X$ and $f_1$ a constant.

Definition 1.12. Let $X$ be a space. A subset $A \subset X$. is called a retract of $X$ iff there is a retraction of $X$ to $A$, i.e. a left inverse to the inclusion $A \to X$, i.e. a map $r : X \to A$ such that $r(a) = a$ for $a \in A$. The subset $A \subset X$. is called a deformation retract of $X$ iff there is a deformation retraction of $X$ to $A$, i.e. a homotopy $\{r_t : X \to X\}_{t \in I}$ such that $r_0(x) = x$ for $x \in X$, $r_1(X) = A$, and $r_t(a) = a$ for $a \in A$ and $t \in I$.

Remark 1.13. If $\{x_0\}$ is a deformation retract of $X$ for some $x_0 \in X$, then certainly $X$ is contractible. A still stronger condition is that $\{x_0\}$ is a deformation retract of $X$ for every $x_0 \in X$. In general the reverse implications do not hold (see Exercises 6 and 7 on page 18 of [3]) but they do hold for cell complexes (see Corollary 0.20 page 16 of [3]). See Section 4 for the definition of cell complex.
Exercise 1.14. Construct explicit homotopy equivalences between the wedge of circles

\[ X = ((-1, 0) + S^1) \cup ((1, 0) + S^1), \]

the spectacles

\[ Y = ((-2, 0) + S^1) \cup ((2, 0) + S^1) \cup \left([-1, 1] \times 0\right), \]

and the space

\[ Z = S^1 \cup \left(0 \times [-1, 1]\right) \]

obtained from a circle by adjoining a diameter.

Solution. Let \( p = (-1, 0), q = (1, 0), \) and \( C \) be the convex hull of \( X \). It suffices to show that each of the spaces \( X, Y, Z \) is homeomorphic to a deformation retract of \( C \setminus \{p, q\} \).

1. The space \( X \) is a deformation retract of \( C \setminus \{p, q\} \). To see this radially project the insides of the two circles from their centers, radially project the portion of \( E \) outside the circles and above the \( x \)-axis from \((0, 2)\), and radially project the portion of \( E \) outside the circles and below the \( x \)-axis from \((0, -2)\).

2. The space \( Y' := \{(x, y) \in \mathbb{R}^2 : (x \pm 1)^2 + y^2 = 1/4 \} \cup ([-1/2, 1/2] \times 0) \) is a deformation retract of \( C \setminus \{p, q\} \). To see this radially project the insides of the two circles from their centers, radially project the points of \( C \) outside the two circles and outside the square \([-1, 1]^2\) from the corresponding center, and project the remaining points vertically.

3. The space \( Z' := \partial C \cup \left(0 \times [-1, 1]\right) \) is a deformation retract of \( C \setminus \{p, q\} \). To see this radially project the left half from \( p \) and the right half from \( q \).

Each fiber of each of these retractions is a closed or half open interval so it is easy to see that each of these retractions is a deformation retraction.

1.15. Suppose that \( E \) is a topological space, \( X \) is a set and \( g : E \to X \) is a surjective map. The \textbf{quotient topology} on \( X \) is defined by the condition that \( U \subset X \) is open if \( g^{-1}(U) \subset E \) is open. That this defines a topology is an immediate consequence of the identities

\[
g^{-1}\left(\bigcup_{\alpha \in \Lambda} U_\alpha\right) = \bigcup_{\alpha \in \Lambda} g^{-1}(U_\alpha), \quad g^{-1}\left(\bigcap_{\alpha \in \Lambda} U_\alpha\right) = \bigcap_{\alpha \in \Lambda} g^{-1}(U_\alpha) \]

for any indexed collection \( \{U_\alpha\}_{\alpha \in \Lambda} \) of subsets of \( X \). Since \( g \) is surjective, \( X \) can be interpreted as the set of equivalence classes of an equivalence relation:

\[ X = E/\sim \quad \text{where} \quad x \sim y \iff g(x) = g(y). \]

In most applications \( X \) is an \textbf{identification space}. Typically

\[ E = Z \sqcup Y, \quad X = Z \sqcup \emptyset Y \]
where \( Y \sqcup Z \) denotes the disjoint union and the notation means that there is a continuous map \( \phi : A \to Z \) with \( A \subset Y \) and \( X = E/\sim \) where the equivalence relation is the minimal equivalence relation satisfying \( y = \phi(x) \implies y \sim x \) for \( x \in A \) and \( y \in Z \). Typically \( Y = D^n \) and \( A = \partial D^n := S^{n-1} \). In this case we say that \( X \) is obtained by attaching an \( n \) cell to \( Z \) along \( \phi \).

**Remark 1.16.** If \( A \subset X \) the notation \( X/A \) indicates the space \( X/\sim \) of equivalence classes where \( x \sim y \iff \) either \( x = y \) or else both \( x \in A \) and \( y \in A \). Typically \( Y = D^n \) and \( A = \partial D^n \). In this case we say that \( X \) is obtained by attaching an \( n \) cell to \( Z \) along \( \phi \).

### 2 Categories

**2.1.** A **category** \( C \) consists of

- a collection called the **objects** of the category;
- for each pair \( X, Y \) of objects a set \( C(X, Y) \) called the morphisms from \( X \) to \( Y \);
- for each object \( X \) a morphism \( \text{id}_X \in C(X, X) \) called the **identity morphism**;
- for each triple \( X, Y, Z \), of objects a map
  \[
  C(X, Y) \times C(Y, Z) \to C(X, Z) : (f, g) \mapsto g \circ f
  \]
  called **composition**;

satisfying

- **(associative law)** \((h \circ g) \circ f = h \circ (g \circ f)\), and
- **(identity laws)** \( \text{id}_Y \circ f = f \circ \text{id}_X = f \) for \( f \in C(X, Y) \).

In most examples the objects are sets with some additional structure and the morphisms are maps which preserve that structure; we write \( f : X \to Y \) instead of \( f \in C(X, Y) \).

**2.2.** A morphism \( g : Y \to X \) is a **left inverse** (resp. **right inverse** to the morphism \( f : X \to Y \)) if \( g \circ f = \text{id}_X \) (resp. \( f \circ g = \text{id}_Y \)). An **isomorphism** is a morphism which has a left inverse and a right inverse. If \( g_1 \) is a left inverse to \( f \) and \( g_2 \) is a right inverse to \( f \), then \( g_1 = g_2 \) and \( g_1 \) is the only left inverse to \( f \) and the only right inverse. Hence an isomorphism \( f : X \to Y \) has a unique **inverse** denoted \( f^{-1} : Y \to X \) characterized by (either of) the equations
\[
 f^{-1} \circ f = \text{id}_X, \quad f \circ f^{-1} = \text{id}_Y.
\]

**2.3.** Here are a few of the categories we will study.
1. The category of groups and group homomorphisms.
2. The category of rings and ring homomorphisms.
3. The category of topological spaces and continuous maps. (The isomorphisms are called homeomorphisms.)
4. The category of topological spaces and homotopy class of maps. (The isomorphisms are called homotopy equivalences.)
5. The category of topological spaces with base point and continuous base point preserving maps \( f : (X, x_0) \to (Y, y_0) \).

2.4. A **covariant functor** from a category \( C \) to a category \( D \) is an operation which assigns to each object \( X \) of \( C \) and object \( \Phi(X) \) of \( D \) and to each morphism \( f \in C(X, Y) \) of \( C \) a morphism \( \Phi(f) \in D(\Phi(X), \Phi(Y)) \) such that

\[
\Phi(id_X) = id_{\Phi(X)}, \quad \Phi(g \circ f) = \Phi(g) \circ \Phi(f).
\]

The notation \( f_* = \Phi(f) \) is often used. A **contravariant functor** from a category \( C \) to a category \( D \) is an operation which assigns to each object \( X \) of \( C \) and object \( \Phi(X) \) of \( D \) and to each morphism \( f \in C(X, Y) \) of \( C \) a morphism \( \Phi(f) \in D(\Phi(Y), \Phi(X)) \) such that

\[
\Phi(id_X) = id_{\Phi(X)}, \quad \Phi(g \circ f) = \Phi(f) \circ \Phi(g).
\]

The notation \( f^* = \Phi(f) \) is often used.

**Example 2.5.** The operation which assigns to each pointed topological space \((X, x_0)\) the fundamental group \( \pi_1(X, x_0) \) is a covariant functor from the category of pointed topological spaces to the category of groups. For \( f : (X, x_0) \to (Y, y_0) \) the induced map \( f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \) sends the homotopy class of a loop \( \gamma \) to the homotopy class of \( f \circ \gamma \).

**Example 2.6.** The operation which assigns to each topological space \( X \) the ring \( C(X, \mathbb{R}) \) of continuous real valued functions on \( X \) is a contravariant functor from the category of topological spaces to the category of rings. For a continuous map \( f : X \to Y \) the induced morphism \( f^* : C(Y, \mathbb{R}) \to C(X, \mathbb{R}) \) is defined by \( f^* u = u \circ f \) for \( u \in C(Y, \mathbb{R}) \).

3 Surfaces

**Exercise 3.1.** Let \( P \) be a polygon with an even number of sides. Suppose that the sides are identified in pairs in any way whatsoever. Prove that the quotient space is a compact surface. A **surface** is a space which is locally homeomorphic to \( \mathbb{R}^2 \). That the sides are identified in pairs means the following. There is an enumeration \( \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \) of the edges of \( P \) (not necessarily in cyclic order but without repetitions) and for each \( k = 1, 2, \ldots, n \) a homeomorphism \( \phi_k : \alpha_k \to \beta_k \) so that desired identification space \( S \) is obtained from \( P \) by identifying \( x \in \alpha_k \subset \partial P \) with \( \phi_k(x) \in \beta_k \subset \partial P \).
Solution. Choose \( p \in S \). We must construct a neighborhood \( U \) of \( p \) in \( S \) and a homeomorphism \( h : U \to \mathbb{R}^2 \). There are three cases.

Case 1. Assume \( p \) lies in the interior of \( P \). Then there is an open disk in \( P \) centered at \( p \). Any open disk is homeomorphic to \( \mathbb{R}^2 \).

Case 2. Assume that \( p = \{a, b\} \) where \( a \in \alpha := \alpha_k \), \( b \in \beta := \beta_k \), and \( \phi(a) = b \) where \( \phi := \phi_k \) for some \( k = 1, 2, \ldots, n \). Abbreviate \( R := (-1, 1) \times (-1, 1) \), \( R_1 := (-1, 1) \times [0, 1) \), \( R_2 := (-1, 1) \times (-1, 1) \), \( I := R_1 \cap R_2 \). Let \( f \) and \( g \) be homeomorphisms from \( R_1 \) and \( R_2 \) onto neighborhoods \( U_1 \) and \( U_2 \) of \( a \) and \( b \) in \( P \) respectively. Thus \( a \in f(I) \subset \alpha \) and \( b \in g(I) \subset \beta \). Shrinking and modifying as necessary we may assume \( \phi \circ f = g \). Define \( \psi : I \to I \) by \( \psi(x) = g^{-1}(\phi(f(x))) \) and \( \Psi : R_1 \to R_1 \) by \( \Psi(x, y) = \psi(x, 0) + (0, y) \). Replacing \( g \) by \( g \circ \Psi \) we may assume that \( \phi \circ f = g \). Now define \( h : R \to S \) by \( h(x, y) = f(x, y) \) if \( y \geq 0 \) and \( h(x, y) = g(x, y) \) if \( y \leq 0 \).

Case 3. Assume that \( p = \{v_1, v_2, \ldots, v_k\} \) is a set of vertices of \( P \). Renaming and replacing some of the \( \phi_j \) by \( \phi_j^{-1} \) we may assume that \( v_j \) is the common vertex of \( \beta_j \) and \( \alpha_j \), where \( \beta_0 := \beta_k \). Let \( R \) be the open unit disk in \( \mathbb{C} = \mathbb{R}^2 \) and write \( R = R_1 \cup \cdots \cup R_k \) where \( R_j \) is the sector

\[
R_j := \left\{ r \exp(i\theta) : 0 \leq r < 1, \quad \frac{2(j-1)\pi}{k} \leq \theta \leq \frac{2j\pi}{k} \right\}.
\]

It is easy to construct homeomorphisms \( f_j : R_j \to U_j \) where \( U_j \) is a neighborhood of \( v_j \) in \( P \), for example we make take \( f_j(r \exp(i\theta)) = v_j + c_j r \exp(i(a_j \theta + b_j)) \) where \( c_j > 0 \) is small and \( a_j \) and \( b_j \) are judiciously chosen. Reflecting if necessary we can even achieve \( f_j(I_{j-1}) \subset \alpha_j \) and \( f_j(I_j) \subset \beta_j \) where

\[
I_j := \left\{ r \exp(i\theta) : 0 \leq r < 1, \quad \theta = \frac{2j\pi}{k} \right\}.
\]

To get a homeomorphism from \( R \) to a neighborhood of \( p \) in \( S \) we must find such homeomorphisms \( f_j \) satisfying the additional condition that

\[
\phi_j \circ f_{j+1}|I_j = f_j|I_j.
\]

For this we try

\[
f_j(r \exp(i\theta)) = v_j + \rho_j(\theta, r) \exp(i(a_j \theta + b_j)).
\]

As in case 2 we modify \( f_{j+1}|I_j \) to achieve \((*)\). This determines \( \rho_j(2(j-1)\pi/k, \cdot) \) and \( \rho_j(2\pi/k, \cdot) \). These last two maps are homeomorphisms (i.e. strictly increasing functions) from from the unit interval to two other intervals and we define \( \rho_j(\theta, \cdot) \) for intermediate values of \( \theta \) by linear interpolation.

Exercise 3.2. Let \( S' \) be the surface obtained by modifying the surface \( S \) of Exercise 3.1 by replacing each homeomorphism \( \phi_k : \alpha_k \to \beta_k \) by a different homeomorphism \( \psi_k : \alpha_k \to \beta_k \). Assume each homeomorphism \( \psi_k^{-1} \circ \phi_k : \alpha_k \to \alpha_k \) fixes the endpoints of the side \( \alpha_k \). Show that \( S \) and \( S' \) are homeomorphic.
Remark 3.3. The following representations are called standard:

(I) $P$ is a square with boundary
\[ \partial P = \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2, \]
the sides are are enumerated and oriented in the clockwise direction, and the identifications reverse orientation.

(II) $P$ is a $4g$-gon with boundary
\[ \partial P = \alpha_1 \cup \alpha_2 \cup \beta_1 \cup \beta_2 \cup \cdots \cup \alpha_{2g-1} \cup \alpha_{2g} \cup \beta_{2g-1} \cup \beta_{2g}, \]
the sides are enumerated and oriented in the clockwise direction, and the identifications reverse orientation.

(III) $P$ is a $2k$-gon with boundary
\[ \partial P = \alpha_1 \cup \beta_1 \cup \cdots \cup \alpha_k \cup \beta_k, \]
the sides are enumerated and oriented in the clockwise direction, and the identifications preserve orientation.

The Classification Theorem for Surfaces (See [2] Page 236, [6] page 9, or [5] page 204) implies that any compact surface is homeomorphic to one of these. One can transform any space $S = P/\sim$ as in Exercise 3.1 to one of the standard representations with a sequence of elementary moves. There are two kinds of elementary moves: one which removes adjacent edges $\alpha$ and $\beta$ that are identified reversing orientation and another which cuts a polygon in two along a diagonal producing a new pair $(\alpha, \beta)$ and then reassembling along an old pair $(\alpha, \beta)$. This process produces a proof of the Classification Theorem from the assertion that any compact surface is homeomorphic to some $P/\sim$ (not necessarily a standard one). The process has nothing to do with proving that $P/\sim$ is a compact surface.

Exercise 3.4. We say surface $S$ is said to be the connected sum of surfaces $S_1$ and $S_2$ and write
\[ S = S_1 \# S_2 \]
iff there are disks $D_i \subset S_i$ ($i = 1, 2$) and a homeomorphism $\phi : \partial D_2 \rightarrow \partial D_1$ with
\[ S = (S_1 \setminus E_1) \sqcup \phi (S_2 \setminus E_2) \]
where $E_i := D_i \setminus \partial D_i$ is the interior of $D_i$ and $\partial D_i$ is the boundary of $D_i$. Show that

1. The connected sum of any surface $S$ with the sphere $S^2$ is homeomorphic to $S$.

2. A surface as in part (II) of Remark 3.3 is homeomorphic to a connected sum
\[ gT^2 := \underbrace{T^2 \# T^2 \# \cdots \# T^2}_g \]
of $g$ tori $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. 
3. A surface as in part (III) of Remark 3.3 is homeomorphic to a connected sum
\[ kP^2 := P^2 \# P^2 \# \cdots \# P^2 \]
of \( k \) projective planes.

**Exercise 3.5.** One says that the surface \( S \# T^2 \) results from \( S \) by adding a **handle** and that the surface \( S \# P^2 \) results from \( S \) by adding a **cross cap**. It is a consequence of the previous exercise that every compact surface is homeomorphic to the surface obtained from a sphere by adding handles and cross caps. Given \( g, k > 0 \), find \( n \) so that \((gT^2) \# (kP^2)\) is homeomorphic to \( nP^2 \).

### 4 Cell Complexes

**4.1.** Throughout this section \( \Phi = \{ \Phi_\alpha : D_\alpha \to X \}_{\alpha \in \Lambda} \) denotes an indexed collection of maps into a topological space \( X \); the domain \( D_\alpha \) of each map is a (space homeomorphic to the standard) closed disk of dimension \( n_\alpha := \dim(D_\alpha) \). Such a collection determines and is determined by a map
\[ \Phi : \bigsqcup_{\alpha \in \Lambda} D_\alpha \to X, \quad \Phi|_{D_\alpha} := \Phi_\alpha \]
from the disjoint union of the disks to \( X \). For each integer \( n \) define
\[ X^{(n)} = X^{(k)}(\Phi) := \bigcup_{n_\alpha \leq n} \Phi_\alpha(D_\alpha). \]

**Definition 4.2.** Such a collection \( \{ \Phi_\alpha \}_{\alpha \in \Lambda} \) as in 4.1 is called a **cell structure** on \( X \) iff it satisfies the following:

1. The restriction of \( \Phi \) to the disjoint union \( \bigsqcup_{\alpha \in \Lambda} (D_\alpha \setminus \partial D_\alpha) \) of the interiors of the disks \( D_\alpha \) is a bijection onto \( X \).
2. \( \Phi(\partial D_\alpha) \subset X^{(n-1)} \) where \( n = n_\alpha \).
3. The space \( X \) has the quotient topology, i.e. a subset \( U \subset X \) is open if and only if \( \Phi^{-1}_\alpha(U) \) is an open subset of \( D_\alpha \) for all \( \alpha \in \Lambda \).

A space \( X \) equipped with a cell structure is called a **cell complex**. The closed subset \( X^{(n)} \subset X \) is called the **n-skeleton**.

**4.3.** The following notations and terminology are used for cell complexes. The sets \( X^{(n)} \) give a **filtration** of \( X \), i.e.
\[ X = \bigsqcup_{n=0}^{\infty} X^{(n)}, \quad X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots . \]
The further notations
\[ e_\alpha := \Phi_\alpha(D_\alpha), \quad e_\alpha := \Phi_\alpha(D_\alpha \setminus \partial D_\alpha), \quad \phi_\alpha := \Phi_\alpha|\partial D_\alpha \]
are commonly used. Thus
\[ X = \bigcup_{\alpha \in \Lambda} e_\alpha, \quad e_\alpha \cap e_\beta = \emptyset \text{ for } \alpha \neq \beta, \]
and each \( e_\alpha \) is homeomorphic to an open disk. The sets \( e_\alpha \) are called cells, the sets \( \bar{e}_\alpha \) are called closed cells, the maps \( \Phi_\alpha \) are called characteristic maps, and the maps \( \phi_\alpha \) are called attaching maps. Sometimes a cell is called an open cell for emphasis, but a cell need not be an open subset of \( X \). It is customary to write \( e_\alpha^n \) to indicate that the dimension of the cell \( e_\alpha \) is \( n = n_\alpha \) so
\[ X^{(n)} = \bigcup_{m \leq n} e_\alpha^m = \bigcup_{m \leq n} \bar{e}_\alpha^m, \quad \text{and} \quad e_\alpha^n \setminus e_\alpha^{n-1} \subset X^{(n-1)}. \]

A cell complex is called finite (resp. countable) iff there are only finitely many (resp. countably many) cells, i.e. iff the index set \( \Lambda \) is finite (resp. countable). A cell complex is called \( n \)-dimensional iff \( X^{(n)} = X \) and \( X^{(n-1)} \neq X \) and is called finite dimensional iff it is \( n \)-dimensional for some \( n \). It is customary to denote a cell complex by the same letter \( X \) as its underlying topological space rather than by the more correct notation \( \{ \Phi_\alpha : D_\alpha \to X \}_{\alpha \in \Lambda} \).

**Example 4.4.** Exercise 3.1 shows how to construct a finite cell complex whose underlying topological space \( X \) is a surface. This cell complex has only one 2-cell. For the surface of type (II) (connected sum of \( g \) copies of the torus) in Remark 3.3 there is one 0-cell, one 2-cell, and \( 2g \) 1-cells, while for the surface of of type (III) (connected sum of \( k \) copies of the projective plane) there is one 0-cell, one 2-cell, and \( k \) 1-cells.

**Exercise 4.5.** Construct an explicit homeomorphism between the sphere \( S^n \) and the cell complex with one 0-cell, one \( n \)-cell, and no other cells.

**Proposition 4.6.** A map \( \Phi \) as in 4.4 with \( \Lambda \) finite and which satisfies (1) and (2) also satisfies (3), i.e. it is a (finite) cell complex.

**Proposition 4.7.** For a cell complex the map \( \Phi : \bigsqcup_{\alpha \in \Lambda} D_\alpha \to X \) is proper, i.e. compact set in a cell complex intersects only finitely many cells of \( X \). In particular, each closed cell \( \bar{e}_\alpha^n \) intersects only finitely many other cells \( \bar{e}_\alpha^m \).

**Proof.** See 3 Proposition A.1 page 520.

**Proposition 4.8.** A subset of a cell complex \( X \) is closed if and only if it intersects each \( n \)-skeleton \( X^{(n)} \) in a closed set.

**Proof.** See 3 Proposition A.2 page 521.
Remark 4.9. Cell complexes were invented by G. H. C. Whitehead [11] who called them CW complexes. He defined a CW complex as a space $X$ equipped with a decomposition $X = \bigcup_{\alpha} e_{\alpha}$ which arises from some cell structure $\{\Phi_{\alpha}\}_{\alpha}$, whereas for us a cell complex is a space $X$ equipped with a cell structure $\{\Phi_{\alpha}\}_{\alpha}$. (Different cell structures can give rise to the same decomposition.) The $C$ stands for closure finite which signifies the conclusion of Proposition 4.7, i.e. that each closed cell $\bar{e}_{n}^{\alpha}$ intersects only finitely many other cells $\bar{e}_{m}^{\beta}$ with $m \leq n$. The $W$ stands for weak topology, the property in the conclusion of Proposition 4.8, that a subset of $X$ is closed if and only if it intersects each $n$-skeleton $X^{(n)}$ in a closed set. Whitehead viewed a CW complex as being constructed inductively: the $n$-skeleton $X^{(n)}$ is constructed from the $(n-1)$-skeleton $X^{(n-1)}$ by attaching the $n$-cells $\bar{e}_{n}^{\alpha}$ along their boundary. This is why the maps $\phi_{\alpha} : \partial D_{n}^{\alpha} \to X^{(n-1)}$ are called attaching maps. The equivalence of Whitehead’s definition with the inductive definition is Proposition A.2 page 521 of [3]. Definition 4.2 is easily seen to be equivalent to the inductive definition. Following [3] we use the terms cell complex and CW complex synonymously, but other authors use the former term to signify a structure satisfying parts (1) and (2) of Definition 4.2 but not necessarily (3). In these notes we are mostly concerned with finite cell complexes where part (3) is superfluous by Proposition 4.6.

Example 4.10. The Hawaiian earring

$$X = \bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 : x^2 - 2n^{-1}x + y^2 = 0\}.$$ 

inherits a topology as a subset of $\mathbb{R}^2$ which is weaker (fewer open sets) than its topology as a countable wedge sum of circles. The former topology satisfies parts (1) and (2) of Definition 4.2 but not (3). The latter topology is its topology as a cell complex. Calling the quotient topology the weak topology is confusing because of this example.

Definition 4.11. A map $f : X \to Y$ between cell complexes is called cellular iff it preserves skeletons, i.e. iff $f(X^{(n)}) \subset Y^{(n)}$ for all $n$. The cell complexes and the cellular maps between them form a category: The identity map of a cell complex is cellular, and the composition of cellular maps is cellular. A cellular isomorphism is an isomorphism in this category, i.e. a cellular map $f : X \to Y$ such that there is a cellular map $g : Y \to X$ satisfying $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Proposition 4.12. A map $f : X \to Y$ between cell complexes is cellular if and only if the image of each cell of $X$ is a subset of a finite union of cells of $Y$ of the same or lower dimension.

Proof. ‘If’ is immediate and ‘only if’ follows from the image of a closed set is compact and therefore a subset of a finite subcomplex of $Y$.

Remark 4.13. A cellular map $f$ need not satisfy the stronger condition that the image of each cell of $X$ is a subset of a cell of $Y$ of the same or lower dimension.
If \( X = Y = I \), \( X^{(0)} = \{0,1\} \), \( Y^{0} = \{0,1/2,1\} \), and \( f \) is the identity map, then \( f(X^{(0)}) \subset Y^{(0)} \) but \( f(X) \) is not contained in a cell of \( Y \). The cell complexes and the cellular maps between them form a category somewhat analogous to the category simplicial complexes and the simplicial maps between them. However, for a simplicial map the image of a simplex is a subset of a simplex of the same or lower dimension.

**Proposition 4.14.** A map \( f : X \to Y \) between cell complexes is a cellular isomorphism if and only if \( f \) is a homeomorphism and maps each cell of \( X \) onto a cell of \( Y \) (necessarily of the same dimension).

**Proof.** A cellular map between cell complexes is continuous so a cellular isomorphism \( f \) is (in particular) a homeomorphism. For each \( k \) map \( f \) induces a homeomorphism \( X^{(k)}/X^{(k-1)} \to Y^{(k)}/Y^{(k-1)} \) and the map \( g := f^{-1} \) induces the inverse homeomorphism. But \( X^{(k)}/X^{(k-1)} \) is a wedge of spheres and becomes disconnected if the wedge point is removed. Since the wedge point of \( X^{(k)}/X^{(k-1)} \) is mapped to the wedge point of \( Y^{(k)}/Y^{(k-1)} \) the components of the complement are preserved, i.e. \( f \) maps each open \( k \)-cell of \( X \) homeomorphically onto a \( k \)-cell of \( Y \) and thus determines a bijection between the \( k \)-cells of \( X \) and the \( k \)-cells of \( Y \). (Homeomorphic cells must have the same dimension by Brouwer’s Invariance of Domain Theorem but that theorem is not used here.)

**Definition 4.15.** A **subcomplex** of a cell complex \( X = \bigcup_{\alpha \in \Lambda} e^n_{\alpha} \) is a closed subset \( A \) of form

\[
A = \bigcup_{\alpha \in A_0} e^n_{\alpha}
\]

where \( A_0 \subset \Lambda \). The condition that \( A \) is closed implies that \( A = \bigcup_{\alpha \in A_0} \overline{e^n_{\alpha}} \). Thus \( A \) inherits a cell structure from \( X \) and the \( k \)-skeleton of \( A \) is \( A^{(k)} = A \cap X^{(k)} \). In particular, the inclusion map \( A \to X \) is cellular. The \( k \)-skeleton \( X^{(k)} \) of \( X \) is a subcomplex. A **cellular pair** is a pair \((X, A)\) consisting of a cell complex \( X \) and a subcomplex \( A \subset X \).

**Remark 4.16.** The definitions do not require that a closed cell be a subcomplex, but this is usually the case.

**Proposition 4.17.** A compact subset of a cell complex lies in a finite subcomplex.

**Proof.** As in Proposition 4.7
point is a zero cell of each). As noted in [3], the product topology is the quotient topology (i.e. Item (3) of Definition [4,2] holds) for countable complexes but this can fail if one of the factors is uncountable.

**Remark 4.19.** There is an example of a product \( X \times Y \) of two cell complexes (with \( X \) countable but \( Y \) not) which is not a cell complex in the product topology.

## 5 Some Homotopies

**Lemma 5.1 (Mailed Fist Homotopy).** Let \((X, A)\) be a cellular pair. Then \((X \times 0) \cup (A \times 1)\) is a deformation retract of \(X \times \mathbb{I}\).

**Proof.** We must show that there is a homotopy \(\{R_t = (r_t, \tau_t) : X \times \mathbb{I} \to X \times \mathbb{I}\}_{t \in \mathbb{I}}\) satisfying \(r_0(x) = x, r_1(a) = a, r_1(A) = A\), and \(\tau_0(x) = 0, \tau_1(a, s) = s\), for \(x \in X, a \in A, t, s \in \mathbb{I}\). See Proposition 0.16 on page 26 of [3].

**Corollary 5.2.** Assume that \((X, A)\) is a cellular pair and that \(A\) is contractible. Then the projection \(X \to X/A\) is a homotopy equivalence.

**Proof.** Define \(K : (X \times 0) \cup (A \times 1) \to (X \times 0) \cup (A \times \mathbb{I})\) by \(K(x, 0) = x\) and \(K(a, t) = k_t(a)\) for \(x \in X, a \in A, t \in \mathbb{I}\). Let \(\{R_t : X \times I \to (X \times 0) \cup (A \times \mathbb{I})\}_{t \in \mathbb{I}}\) be as in the proof of Lemma 5.1. Then \(R_1 : X \times I \to (X \times 0) \cup (A \times I)\) is a retraction, i.e. \(R_1(x, 0) = (x, 0)\) and \(R_1(a, t) = (a, t)\). Define \(f : X \to X\) by \(f_t(x) = K(R_t(x, t))\) so \(f_0(x) = K(x, 0) = x\) and \(f_1(x) = K(a, 1) = k_1(a) = a_0\). Let \(\pi : X \to X/A\) be the projection and define \(\sigma : X/A \to X\) by \(\sigma(x) = f_1(x)\) for \(x \in X \setminus A\) and \(\sigma(A) = a_0\). Then \(\sigma \circ \pi = f_1 \simeq \text{id}_X\) and \(\pi \circ \sigma \simeq \text{id}_{X/A}\) by the homotopy \(h_t(x) = \pi(f_t(x))\) for \(x \in X \setminus A\) and \(h_0(A) = \{A\}\).

**Corollary 5.3.** Assume that \((X, A)\) is a cellular pair and that \(f, g : A \to Y\) are homotopic. Then \(Y \cup_f X \simeq Y \cup_g X\).

**Proof.** By hypothesis there is a map \(H : A \times \mathbb{I} \to Y\) with \(H(\cdot, 0) = f\) and \(H(\cdot, 1) = g\). There are inclusions

\[
\iota_0 : Y \cup_f X \to Y \cup_H (X \times \mathbb{I}), \quad \iota_1 : Y \cup_g X \to Y \cup_H (X \times \mathbb{I})
\]

induced from the inclusions \(X \times 0 \subseteq X \times \mathbb{I}\) and \(X \times 1 \subseteq X \times \mathbb{I}\); it is enough to show that \(\iota_0\) and \(\iota_1\) are homotopy equivalences. We prove the former; the same argument (read \(1 - t\) for \(t\)) proves the latter.

**Step 1.** We define \(F : Y \cup_H (X \times \mathbb{I}) \to Y \cup_f X\). Let \(\{R_t\}_{t \in \mathbb{I}}\) be deformation retraction from \(X \times \mathbb{I}\) onto \((X \times 0) \cup (A \times \mathbb{I})\) as in Lemma 5.1 and let \(R_t = (r_t, \tau_t)\). Define \(F : Y \cup_H (X \times \mathbb{I}) \to Y \cup_f X\) by \(F(y) = y\) for \(y \in Y\) and

\[
F(x, s) = \begin{cases} 
H(R_1(x, s)) \in Y & \text{if } R_1(x, s) \in A \times \mathbb{I}, \\
 r_1(x) \in X & \text{if } R_1(x, s) \in X \times 0.
\end{cases}
\]
The map \( F \) is well defined as \( H(R_1(x,s)) = f(r_1(x)) \sim r_1(x) \) for \( R_1(x,s) \in A \times 0 \).

**Step 2.** We show \( F \circ t_0 \simeq \text{id}_{Y \cup_f X} \). Define \( \{ \Phi_t : Y \cup_f X \to Y \cup_f X \}_{t \in I} \) by \( \Phi_t(y) = y, \Phi_t(x) = r_t(x) \). The map \( \Phi_t \) is well defined because \( r_t(x) = x \sim f(x) \) for \( x \in A \). One easily checks that \( \Phi_0 = \text{id}_{Y \cup_f X} \) and \( \Phi_1 = F \circ t_0 \).

**Step 3.** We show \( t_0 \circ F \simeq \text{id}_{Y \cup_f (X \times I)} \). Define \( \{ \Psi_t : Y \cup_f H \times Y \cup_f H \times Y \}_{t \in I} \) by \( \Psi_t(x) = y \) for \( y \in Y \) and \( \Psi_t(x,s) = R_t(x,s) \in X \times I \) for \( (x,s) \in X \times I \). The map \( \Psi_t \) is well defined because \( R_t(x,s) = (x,s) \sim H(x,s) \) for \( (x,s) \in A \times I \).

One easily checks that \( \Psi_0 = \text{id}_{Y \cup_f (X \times I)} \) and \( \Psi_1 = t_0 \circ F \).

**Remark 5.4.** Let \( Y = A, f = \text{id}_A : A \to Y \) and \( g : A \to Y \) be a constant map: \( g(a) = a_0 \). Clearly the space \( Y \cup_f X \) is homeomorphic to the space \( X \) and the space \( Y \cup_g X \) is homeomorphic to the wedge sum \( A \vee (X/A) \) where the point \( a_0 \in A \) is identified. Hence if \( A \) is contractible, then \( f \) and \( g \) are homotopic so \( X \) and \( A \vee (X/A) \) are homotopy equivalent. By Corollary 5.6 below, \( \{ a_0 \} \) is a deformation retract of \( A \). Hence the wedge sum \( A \vee (X/A) \) is homotopy equivalent to \( X/A \). Thus Corollary 5.2 is a consequence of Corollary 5.3.

**Theorem 5.5.** Let \( (X, A) \) be a cellular pair. Then the inclusion map \( \iota : A \to X \) is a homotopy equivalence if and only if \( A \) is a deformation retract of \( X \).

**Proof.** (This is Corollary 0.20 page 16 of [3].) The following proof is from [12] page 25.) 'If' is trivial. For 'only if' let \( f : X \to A \) be a homotopy inverse to \( \iota \). Let \( \{ \phi_t : A \to A \}_{t \in I} \) be a homotopy with \( \phi_1 = \text{id}_A \) and \( \phi_0 = f \circ \iota \). Extend to a homotopy \( \{ \Phi_t : X \to A \}_{t \in I} \) with \( \Phi_0 = f \). Then \( \Phi_1 : X \to A \) is a retraction. Hence, replacing \( f \) by \( \Phi_1 \), we may assume w.l.o.g. that the homotopy inverse \( f \) to \( \iota \) is a retraction, i.e. that \( f|A = \text{id}_A \). Let \( \{ \psi_t : X \to X \}_{t \in I} \) be a homotopy from \( \psi_0 = \text{id}_X \) to \( \psi_1 = \iota \circ f \). Let \( (Y, B) \) be the cellular pair defined by

\[
Y = I \times X, \quad B = (\{0, 1\} \times X) \cup (I \times A).
\]

Define \( h : (\{0\} \times Y) \cup (I \times B) \to X \) by

\[
\begin{align*}
    h(0, t, x) &= \psi_t(x) \\
    h(s, 0, x) &= \psi_0(x) = x \\
    h(s, t, a) &= \psi_{(1-s)t}(a) \\
    h(s, 1, x) &= \psi_{1-s}(f(x))
\end{align*}
\]

for \( s, t \in I, x \in X, \) and \( a \in A \). By Lemma 5.1 extend \( h \) to \( H : I \times Y \to X \) and define \( \{ g_t : X \to X \}_{t \in I} \) by \( g_t(x) = H(1, t, x) \). Then

\[
g_t|A = \text{id}_A, \quad g_0 = \text{id}_X, \quad g_1 = f,
\]

as \( g_t(a) = H(1, t, a) = h(1, t, a) = \psi_0(a) = a, \)
(\( a \in A \)) \( g_0(x) = H(1, 0, x) = h(1, 0, x) = \psi_0(x) = x, \) and \( g_1(x) = H(1, 1, x) = \psi_0(f(x)) = f(x), \) for \( a \in A \) and \( x \in X \).}

**Corollary 5.6.** For a cell complex \( X \) the following are equivalent:
(i) $X$ is contractible.

(ii) For some $a_0 \in X$, $\{a_0\}$ is a deformation of $X$.

(iii) For every $a_0 \in X$, $\{a_0\}$ is a deformation of $X$.

Remark 5.7. Of course, the implications (iii) $\implies$ (ii) $\implies$ (i) in Corollary 5.6 hold for any space. Exercises 6 and 7 on page 18 of [3] show that the implications (ii) $\implies$ (iii) and (i) $\implies$ (ii) do not hold for general spaces.

Proposition 5.8. Let $(X, A)$ be a cellular pair. Then there is a neighborhood $N$ of $A$ in $X$ such that $A$ is a deformation retract of $N$.

Proof. This is Proposition A.5 page 523 of [3].

Corollary 5.9. A cell complex is locally contractible, i.e., for every $x_0 \in X$ and every neighborhood $V$ of $x_0$ there is a neighborhood $U \subset V$ of $x_0$ and a deformation retraction of $U$ onto $\{x_0\}$.

Problem 5.10. It is plausible that a retraction whose fibers are (homeomorphic to) intervals is a deformation retraction, but I doubt that this is true in complete generality. However, a theorem of Smale [9] combined with Whitehead’s Theorem (Theorem 4.5 page 346 of [3]) implies that map between finite cell complexes with contractible fibers is a homotopy equivalence. It then follows from Theorem 5.5 that if $(X, A)$ is a cellular pair and $r : X \to A$ is a retraction with contractible fibers, then $A$ is a deformation retract of $X$. Prove (or disprove) the following slightly stronger statement: If $(X, A)$ is a cellular pair then a retraction $r : X \to A$ with contractible fibers is a deformation retraction.

Exercise 5.11. Let $X = D^2 \times I$ be a solid cylinder at $\alpha : I \to X$ be a polygonal arc, possibly knotted, with $\alpha(0) = (0, 0, 1) \in D^2 \times 1$ and $\alpha(1) = (0, 0, 0) \in D^2 \times 0$. Show that the image $A := \alpha(I)$ is a deformation retract of $X$. (This example comes from Conley [1] page 26.)

Exercise 5.12. Given positive integers $v, e, f$ satisfying $v - e + f = 2$ construct a cell complex homeomorphic to $S^2$ having $v$ 0-cells, $e$ 1-cells, and $f$ 2-cells.

Solution. If $v - e + f = 2$ then

$$(v, e, f) = (1, 0, 1) + m(1, -1, 0) + n(0, -1, 1)$$

where $m = v - 1$ and $n = f - 1$. The sphere is a cell complex with $v = f = 1$ and $e = 0$ in the only way possible: the attaching map is constant. To increase $m$ by one without changing $n$ add a new vertex and edge and modify the corresponding characteristic map by composing with complex square root. To increase $n$ without changing $m$ insert a loop at a vertex. The result follows by induction. 

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6 Δ-Complexes and Simplicial Complexes

Definition 6.1. Resume the notation of [1,1] and Definition [4,2]. A cell complex \( \{ \Phi_\alpha : D_\alpha \to X \}_{\alpha \in \Lambda} \) is called a Δ-complex iff each component \( D_\alpha \) is a standard \( n \) simplex

\[
\Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{I}^{n+1} : x_0 + x_1 + \cdots + x_n = 1\}
\]

(as before, \( n = n_\alpha \) depends on \( \alpha \)) and for each face \( \iota : \Delta^{n-1} \to \Delta^n \) and each characteristic map \( \Phi_\alpha \) the composition \( \Phi_\alpha \circ \iota \) is also a characteristic map. The term face here means

\[
\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, x_{k+1}, \ldots, x_n)
\]

for some \( k = 0, \ldots, n \), i.e. \( \iota = \iota_k \) as in [1,3]. The points of the 0-skeleton of a Δ-complex are called vertices and the closed cells \( e^n_\alpha := \Phi_\alpha(\Delta^n) \) are called simplices. A Δ-complex is called a simplicial complex iff the characteristic maps are determined by the vertices, i.e. whenever \( \alpha, \beta \in \Lambda \) satisfy \( (n_\alpha = n_\beta \) and \( \Phi_\alpha(v) = \Phi_\beta(v) \) for each of the \( n+1 \) vertices \( v = (0, \ldots, 0, 1, 0, \ldots, 0) \) of \( \Delta^n \), we have \( \alpha = \beta \). A simplicial complex \( \{ \Phi_\alpha : D_\alpha \to X \}_{\alpha \in \Lambda} \) is called a triangulation of \( X \). As was the case with cell complexes, it is customary to denote a Δ-complex by the same letter \( X \) as its underlying space rather than by the more correct notation \( \{ \Phi_\alpha : D_\alpha \to X \}_{\alpha \in \Lambda} \).

Example 6.2. The simplex \( \Delta^n \) is a simplicial complex with a characteristic map \( \Phi_\alpha : \Delta^m \to \Delta^n \) for every subset \( \alpha \subset \{0, 1, \ldots, n\} \). The map \( \Phi_\alpha \) is the map

\[
\Phi_\alpha(x) = \sum_{j=0}^{m} x_j v_{ij}
\]

for \( x = (x_0, x_1, \ldots, x_m) \in \Delta^m \), \( \alpha = \{i_0, i_1, \ldots, i_m\} \) with \( i_0 < i_1 < \cdots < i_m \) and where \( v_i \) is the vertex of \( \Delta^n \) with 1 in the \( i \)th place and 0 elsewhere.

Example 6.3. The torus \( T^2 \) is the surface obtained from the unit square \( I^2 \) with the identifications \( (x, 0) \sim (x, 1) \) and \((0, y) \sim (1, y) \). The Klein bottle \( K^2 \) is the surface obtained from the unit square \( I^2 \) with the identifications \( (x, 0) \sim (x, 1) \) and \((0, y) \sim (1, 1-y) \). The projective plane \( P^2 \) is the surface obtained from the unit square \( I^2 \) with the identifications \( (x, 0) \sim (1-x, 1) \) and \((0, y) \sim (1, 1-y) \). For \( X = T^2, K^2, P^2 \) there is a Δ-complex \( \Phi \) as indicated in Figure [1]. For \( X = T^2 \) and \( X = K^2 \) there are one 0-simplex \( v \), three 1-simplices \( a, b, c \), and two 2-simplices \( U \) and \( L \). For \( X = P^2 \) we must adjoin another 0-simplex \( w \). In each case the bijections \( \Phi_\alpha \) is determined by the diagram in the only way possible so that the corresponding maps to \( I^2 \) are affine and the arrows on the edges match up. Note that had we reversed the arrows marked \( a \) in \( P^2 \) the vertices of triangle \( U \) would be cyclically (not linearly) ordered. The figure would still represent the projective plane but not a Δ-complex.
Example 6.4. Let $T^2 = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}^2 / \mathbb{Z}^2$ be the torus. There is a cell complex $\Phi$ with one 0-cell (the origin (0, 0) in $\mathbb{R}^2$), two 1-cells (the intervals $\mathbb{I} \times \{0\}$ and $\{0\} \times \mathbb{I}$), and one 2-cell (the unit square $\mathbb{I} \times \mathbb{I}$); the characteristic map is $\Phi(x, y) = (x, y) \mod \mathbb{Z}^2$. If we add the diagonal we get the $\Delta$-complex of Example 6.3. There is an easy triangulation with vertices $(p/3, q/3) \mod \mathbb{Z}^2$ for $p, q = 0, 1, 2$ and edges from $(p/3, q/3)$ to $(p + 1)/3, (q/3)$, $(p/3, (q + 1)/3)$, and $((p + 1)/3, (q + 1)/3)$. This triangulation has 9 vertices, 27 edges, and 18 faces. By judiciously discarding a few vertices and edges we can achieve a triangulation with 7 vertices. (See Exercise 6.6.)

6.5. It is easy to construct simplicial complexes whose geometric realizations are homeomorphic to the sphere $S^2$, the cylinder $S^1 \times I$, the Möbius band, the torus, the Klein bottle, and the projective plane. (See [7] pages 16-19.)

Exercise 6.6. Show that for any triangulation of a compact surface, we have $3f = 2e$, $e = 3(v - \chi)$, and $v \geq (7 + \sqrt{49 - 24\chi})/2$. In the case of the sphere, projective plane and torus, what are the minimum values of the numbers $v$, $e$ and $f$? (Here $v, e$ and $f$ denote the number of vertices, edges and triangles respectively; $\chi := v - e + f$ denotes the Euler characteristic.)

Solution. Let $P$ be the set of pairs $(E, F)$ such that $F$ is a face and $E$ is an edge in the boundary of $F$. Then $P_E = \{ F : (E, F) \in P \}$, and $P_F = \{ E : (E, F) \in P \}$. Then $\#(P_E) = 2$ for each $E$ so $\#(P) = 2e$. Also $\#(P_F) = 3$ for each $F$ so $\#(F) = 3f$. Hence $2e = 3f$. From $v - e + f = \chi$ we get $e = 3v - 3\chi$. But clearly $e \leq \binom{v}{2} = \frac{v(v-1)}{2}$ so $0 \leq v^2 - 7v + 6\chi$ so $v \geq \frac{7 + \sqrt{49 - 24\chi}}{2}$. For the sphere $S^2$ we have $\chi = 2$ and hence $v \geq 4$ with equality for the tetrahedron. For the projective plane we have $\chi = 1$ so $v \geq 6$ with equality for the triangulation shown on page 15 of [6]. (This example is obtained by identifying opposite edges of a hexagon and adding a triangle in the interior.) For the torus $T^2$ we have $\chi = 0$ so $v \geq 7$ with equality in the following triangulation:

![Figure 1: Three $\Delta$-complexes](image-url)
Note that in all cases the lower bound is achieved when the 1-skeleton is the complete graph on $v$ vertices. It is a consequence of the proof that this complete graph is the 1-skeleton of a surface of Euler characteristic $\chi$ only when the lower bound is an integer.

6.7. Let $\{\Phi_\alpha : D_\alpha \to X\}_{\alpha \in \Lambda}$ and $\{\Psi_\beta : D_\beta \to Y\}_{\beta \in \Lambda'}$ be $\Delta$-complexes. A map $f : X \to Y$ is called $\Delta$-map iff for every $\alpha \in \Lambda$ there is a (necessarily unique) $\beta \in \Lambda'$ and a simplicial map $f_\alpha : D_\alpha \to D_\beta$ such that

$$\Psi_\beta \circ f_\alpha = f \circ \Phi_\alpha.$$ 

That a map $g : \Delta^n \to \Delta^m$ is simplicial means that it sends each vertex $v_i$ of $\Delta^n$ to a vertex $g(v_i)$ of $\Delta^m$ and that it is affine, i.e.

$$g \left( \sum_{i=0}^{n} x_i v_i \right) = \sum_{i=0}^{n} x_i g(v_i).$$

With respect to the simplicial structure of Example 6.2 a map $g : \Delta^n \to \Delta^m$ is simplicial if and only if it is a $\Delta$-map. A $\Delta$-map between simplicial complexes is also called a simplicial map.

6.8. The $\Delta$-complexes and $\Delta$-maps between them form the objects and morphisms of a category, i.e, the identity map of of a $\Delta$-complex is a $\Delta$-map and the composition of two $\Delta$-maps is a $\Delta$-map. The simplicial complexes form a (full) subcategory.

6.9. If $(X, A)$ is a cellular pair and $X$ is a $\Delta$-complex, then the subcomplex $A$ is also a $\Delta$-complex; in this case we call $(X, A)$ a $\Delta$-pair. The cell structure on $X/A$ of 4.18 is then a $\Delta$-complex and the projection $X \to X/A$ is a $\Delta$-map. (This construction does not give a simplicial complex even when $X$ is a simplicial complex.) As in 4.18 the operations of suspension, cone, join, and product also yield $\Delta$-complexes (resp. simplicial complexes) when applied to $\Delta$-complexes (resp. simplicial complexes). The characteristic maps of the join are determined by composition with the inverse of the canonical homomorphism

$$\Delta^m \ast \Delta^n \to \Delta^{m+n+1} : (x, y, t) \mapsto tx + (1-t)y.$$ 

The characteristic maps of the product $\Delta$-complex are given by composition with the maps

$$\ell_\sigma : \Delta^{m+n} \to \Delta^m \times \Delta^n.$$
described on page 277 of [3]. In the special case \( n = 1 \) this reduces to the prism structure on \( \Delta^m \times I \) described in the proof of Theorem 2.10 on page 111 of [3]. As is the case for cell complexes, the product topology is the quotient topology for countable complexes but this can fail if one of the factors is uncountable.

7 Abstract Simplicial Complexes*

7.1. An abstract simplicial complex is a collection \( K \) of nonempty finite sets called simplices such that every nonempty subset of a simplex is a simplex, i.e. \( \emptyset \neq \tau \subset \sigma \in K \implies \tau \in K \).

A subset of a simplex is called a face of the simplex. The elements of the set \( V := \bigcup_{\sigma \in K} \sigma \) are called vertices. Since \( v \in V \iff \{v\} \in K \) it is customary not to distinguish between \( v \in V \) and \( \{v\} \in K \). A subset of \( K \) that is itself a simplicial complex is called a subcomplex. If \( K_1 \) and \( K_2 \) are simplicial complexes with vertex sets \( V_1 \) and \( V_2 \), a map \( f : K_1 \to K_2 \) is called a simplicial map if there is a (necessarily unique) map \( f : V_1 \to V_2 \) denoted by the same letter such that

\[
 f(\{v_0, v_1, \ldots, v_k\}) = \{f(v_0), f(v_1), \ldots, f(v_k)\}
\]

for \( \{v_0, v_1, \ldots, v_k\} \in K_1 \). Abstract simplicial complexes and simplicial maps form the objects and morphisms of a category.

7.2. The dimension \( \dim(\sigma) \) of a simplex \( \sigma \) is one less than its cardinality. A simplex of dimension \( k \) is also called a \( k \)-simplex. The subcomplex

\[
 K^{(k)} := \{ \sigma \in K : \dim(\sigma) \leq k \}
\]

is called the \( k \)-skeleton of the simplicial complex \( K \). As noted above we may denote the set of vertices by

\[
 V = K^{(0)}.
\]

A simplicial complex is called \( n \)-dimensional if \( K^{(n)} = K \) and \( K^{(n-1)} \neq K \); it is called finite dimensional if it is \( n \)-dimensional for some \( n \in \mathbb{N} \). A simplicial complex is called finite if the set \( K \) is itself finite.

7.3. The geometric realization of an abstract simplicial complex \( K \) is the set

\[
 |K| := \left\{ x \in [0,1]^V \mid \sum_{v \in V} x(v) = 1 \text{ and } \text{supp}(x) \in K \right\}.
\]

Here \( V \) is the vertex set of \( K \) and \( \text{supp}(x) := \{ v \in V : x(v) \neq 0 \} \) is the support of the element \( x \in |K| \). A simplicial map \( f : K \to L \) induces a map \( f : |K| \to |L| \) called the geometric realization of \( f \) via the formula

\[
 f(x)(w) = \sum_{f(v) = w} x(v)
\]
for \( x \in |K| \). Geometric realization is functorial.

**Theorem 7.4.** Assume that \( K \) is a finite abstract simplicial complex of dimension \( n \). Let \( E = \mathbb{R}^{2n+1} \) denote the Euclidean space of dimension \( 2n+1 \) and \( E^V \) denote the space of all maps from \( f : V \to E \). (The space \( E^V \) has dimension \( \#(V) \cdot (2n+1) \).) Call an element \( f \in E^V \) **generic** iff whenever \( v_0, v_1, \ldots, v_k \) are distinct points of \( V \) and \( k \leq 2n+1 \) the vectors \( f(v_1) - f(v_0), \ldots, f(v_k) - f(v_0) \) are linearly independent. Then

(i) The set of generic elements is open and dense in \( E^V \).

(ii) If \( f \) is generic, the induced map \( f : |K| \to E \) defined by

\[
    f(x) := \sum_{v \in V} x(v)f(v)
\]

is injective.

(iii) The topology on \( |K| \) induced by this embedding is independent of the choice of the generic element \( f \) used to define it.

(iv) In this topology the geometric realization of a simplicial map between finite complexes is continuous.

**Proof.** For a sequence \( W = (v_0, v_1, \ldots, v_k) \) of distinct elements of \( V \) of form the matrix \( A_W(f) \in \mathbb{R}^{(2n+1) \times k} \) whose columns are the \( k \) vectors \( f(v_i) - f(v_0) \) for \( i = 1, \ldots, k \). For each \( k \) element subset \( R \subset \{1, \ldots, 2n+1\} \) let \( a^R_W(f) \) be the determinant of the square matrix that results from \( A_W(f) \) by deleting the rows whose index is not in \( R \). Then \( f \) is generic if and only if for every \( W \) there is an \( R \) such that \( a^R_W(f) \neq 0 \). Since \( a^R_W(f) \) is a polynomial in the entries of \( f \) it cannot vanish identically on an open subset of \( f \)'s. This proves (i). Since \( |K| \) is compact, the map \( f : |K| \to E \) is an embedding if and only if it is injective. But a generic map must be injective: if \( f(x) = f(y) \) then (as the supports of \( x \) and \( y \) are of cardinality at most \( n+1 \)) the union \( W \) of the supports of \( x \) and \( y \) has cardinality at most \( 2n+2 \). But the equation \( f(x) = f(y) \) is a linear relation between the vectors \( \{f(v) : v \in W\} \). As these vectors are independent we must have \( x = y \) as required.

**Remark 7.5.** The induced map \( f : |K| \to E \) can be injective even when \( f \) is not generic. For countable \( K \) of dimension \( n \), item (i) of Theorem 7.4 continues to hold provided open and dense is weakened to residual (countable intersection of open dense sets) so generic maps \( f \) exist by the Baire Category Theorem. Item (ii) will hold but item (iv) is false. As a subset of \( [0,1]^V \) the geometric realization inherits a topology from the product topology of \( [0,1]^V \). In this topology a set \( A \subset |K| \) is closed if and only if \( A \cap |\sigma| \) is closed for every \( \sigma \in K \), but (in the infinite case) this topology can be different from the one induced by an embedding in \( E \).
7.6. An ordered simplicial complex is a simplicial complex equipped with a linear ordering for each simplex in such a way that if \( \tau \) is a face of \( \sigma \) then the ordering assigned to \( \tau \) is the restriction of the ordering assigned to \( \sigma \). A simplicial complex may be ordered in many ways. Generally we will choose an ordering for book keeping purposes and then prove that the choice doesn’t matter.

Remark 7.7. The geometric realization \(|K|\) of an ordered simplicial complex is a \( \Delta \)-complex

\[
\Phi := \{ \Phi_\sigma : \Delta^{\dim(\sigma)} \to |\sigma| \}_{|\sigma|}
\]

via

\[
\Phi_\sigma(x) = x_0 \delta_{v_0} + x_1 \delta_{v_1} + \cdots + x_n \delta_{v_n}
\]

for \( x = (x_0, x_1, \ldots, x_n) \in \Delta^n \) (\( n = \dim(\sigma) \)) and \( \sigma = \{v_0, v_1, \ldots, v_n\} \) where \( v_0 < v_1 < \cdots v_n \).

Problem 7.8. An abstract unordered \( \Delta \)-complex is a disjoint union

\[
W = \bigsqcup_{\alpha \in \Lambda} \sigma_{\alpha},
\]

of nonempty finite subsets together with a function which assigns to each pair \( (w, \alpha) \) with \( \#(\sigma_{\alpha}) > 1 \) a bijection \( \iota : \sigma_{\beta} \to \sigma \setminus \{w\} \). (The index \( \beta \in \Lambda \) is also a function of \( (\alpha, w) \).) The elements of \( W \) are called vertices, the sets \( \sigma_{\alpha} \) are called simplices, and the maps \( \iota \) are called attaching maps. An abstract ordered \( \Delta \)-complex is an abstract unordered \( \Delta \)-complex together with a linear ordering for each simplex \( \sigma_{\alpha} \) such that each attaching map \( \iota \) is the unique order isomorphism from its domain to its target. Does every abstract unordered \( \Delta \)-complex arise from an abstract ordered \( \Delta \)-complex? (In other words given an abstract unordered \( \Delta \)-complex can we order its simplices so that for each \( (w, \alpha) \) the bijection \( \iota \) is the unique order isomorphism?)

Remark 7.9. An abstract \( \Delta \)-complex determines a cell complex \( \Phi : D \to X \) where

\[
D := \bigsqcup_{\alpha \in \Lambda} |\sigma_{\alpha}|
\]

is the disjoint union of the geometric realizations of the simplices, \( X = D/\sim \), and \( \Phi : D \to X \) is the projection into the identification space. The equivalence relation is generated by

\[
x = \sum_{v \in \beta} x_v \delta_v \in |\sigma_{\beta}| \quad y = \sum_{v \in \beta} x_v \delta_{\iota(v)} \in |\sigma_{\alpha}| \quad \implies x \sim y.
\]

This cell complex is called the geometric realization of the abstract \( \Delta \)-complex.
References


