Differential Forms

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1 Algebra

1.1. For (real) vector spaces V and W denote by $L(V, W)$ the vector space of linear maps from V to W and by $L^k(V, W)$ the space of k-**multilinear** maps from V^k to W:

$$L^0(V, W) := W, \qquad L^{k+1}(V, W) := L(V, L^k(V, W)).$$

Denote by $\Lambda^k(V) \subset L^k(V, \mathbb{R})$ the subspace of **alternating** (also called **skew symmetric**) multilinear maps. This means that for $\omega \in L^k(V, \mathbb{R})$ we have

$$\omega \in \Lambda^k(V) \iff \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega(v_1, \ldots, v_k)$$

for all $v_1, v_2, \ldots, v_k \in V$ and all permutations σ of $\{1, 2, \ldots, k\}$. In particular, $\Lambda^0(V) = \mathbb{R}$ and $\Lambda^1(V) = L^1(V, \mathbb{R}) = L(V, \mathbb{R}) = V^*$ the dual vector space of V. A linear map $A \in L(V, W)$ induces a linear map $A^* : \Lambda^k(W) \to \Lambda^k(V)$ via

$$A^*\eta(v_1, v_2, \ldots, v_k) = \eta(Av_1, Av_2, \ldots, Av_k)$$

for $\eta \in \Lambda^k(W)$ and $v_1, v_2, \ldots, v_k \in V$.

1.2. The **wedge product** is the bilinear operation

$$\Lambda^{k_1}(V) \times \Lambda^{k_2}(V) \to \Lambda^{k_1+k_2}(V) : (\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2$$

defined by

$$\omega_1 \wedge \omega_2(v_1, \ldots, v_{k_1+k_2}) = \sum_{\sigma} \operatorname{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \ldots v_{\sigma(k_1)})\omega_2(v_{\sigma(k_1+1)}, \ldots v_{\sigma(k_1+k_2)})$$

where the sum is over all (k_1, k_2)-shuffles. (A (k_1, k_2)-**shuffle** is a permutation σ of $\{1, \ldots, k_1+k_2\}$ such that $\sigma(1) < \cdots < \sigma(k_1)$ and $\sigma(k_1+1) < \cdots < \sigma(k_1+k_2)$.)

1.3. The **interior product** is the bilinear operation

$$V \times \Lambda^k(V) \to \Lambda^{k-1}(V) : (v, \omega) \mapsto \iota(v)\omega$$

defined by

$$\iota(v)\omega(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}).$$

We define $\iota(v)\omega = 0$ when $k = 0$.

Theorem 1.4. *The wedge product is associative and skew commutative, the interior product is a skew derivation, and* $(\Lambda^*(V), \iota(v))$ *is a chain complex, i.e.*

$$\begin{aligned}
(\omega_1 \wedge \omega_2) \wedge \omega_3 &= \omega_1 \wedge (\omega_2 \wedge \omega_3) \\
\omega_2 \wedge \omega_1 &= (-1)^{k_1 k_2} \omega_1 \wedge \omega_2 \\
\iota(v)\big(\omega_1 \wedge \omega_2\big) &= \big(\iota(v)\omega_1\big) \wedge \omega_2 + (-1)^{k_1}\big(\omega_1 \wedge (\iota(v)\omega_2\big) \\
\iota(v)^2 &= 0
\end{aligned}$$

for $\omega_1 \in \Lambda^{k_1}(V)$, $\omega_2 \in \Lambda^{k_2}(V)$, $\omega_3 \in \Lambda^{k_3}(V)$, *and* $v \in V$.

2 Calculus on $\mathbb{R}^n$

2.1. Let $f : X \to V$ be a smooth function mapping an open set $X \subset \mathbb{R}^m$ to a vector space V. We use the following notation for derivatives:

$$Df(x)v := \frac{d}{dt} f(c(t))\bigg|_{t=0}$$

where $c : \mathbb{R} \to X$ is any curve satisfying $c(0) = x \in X$ and $\dot{c}(0) = v \in \mathbb{R}^m$. (By the chain rule the result is independent of the choice of c satisfying these two conditions and is linear in v.) Then $Df : X \to L(\mathbb{R}^m, V)$ so we may define inductively

$$D^{k+1}f := D(D^k f) : X \to L(\mathbb{R}^m, L^k(\mathbb{R}^m, V)) = L^{k+1}(\mathbb{R}^m, V).$$

The kth derivative is symmetric, i.e.

$$D^k f(x) v_1 v_2 \dots v_k = D^k f(x) v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(k)}$$

for any permutation σ of $\{1, 2, \dots, k\}$.

2.2. For an open subset $X \subset \mathbb{R}^m$ denote by

$$\mathcal{X}(X) := C^\infty(X, \mathbb{R}^m)$$

the space of smooth vector fields $v : X \to \mathbb{R}^m$, and by

$$\Omega^k(X) := C^\infty(X, \Lambda^k(\mathbb{R}^m))$$

the space of smooth differential k-forms $\omega : X \to \Lambda^k(\mathbb{R}^m))$ on X. A smooth map $f : X \to Y$ from $X \subset \mathbb{R}^m$ to $Y \subset \mathbb{R}^n$ determines a map $f^* : \Omega^k(Y) \to \Omega^k(X)$ via

$$(f^*\eta)(x)(v_1, \dots, v_p) := \eta(f(x))(Df(x)v_1, \dots, Df(x)v_p).$$

Define the operations

$$\Omega^{k_1}(X) \times \Omega^{k_2}(X) \to \Omega^{k_1+k_2}(X) : (\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2$$

and

$$\mathcal{X}(X) \times \Omega^p(V) \to \Omega^{p-1}(X) : (v, \omega) \mapsto \iota(v)\omega$$

pointwise, i.e.

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x)$$

and

$$(\iota(v)\omega(x)) = \iota(v(x))\omega(x)$$

for $x \in X$.

Theorem 2.3. *Let $X \subset \mathbb{R}^m$ be open. There is a unique operation*

$$d : \Omega^*(X) \to \Omega^{*+1}(X)$$

called the **exterior derivative** *satisfying the following axioms.*

(1) *The exterior derivative $dh \in \Omega^1(X)$ of a 0-form $h \in \Omega^0(X)$ is the ordinary derivative, i.e.*

$$dh = Dh.$$

Note that $\Omega^0(X) = C^\infty(X, \mathbb{R})$ and $\Omega^1(X) = C^\infty(X, \mathbb{R}^{m*})$.

(2) *The exterior derivative is an antiderivation, i.e.*

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge (d\omega_2).$$

for $\omega_1 \in \Omega^{k_1}(X)$ and $\omega_2 \in \Omega^{k_2}(X)$.

(3) *The pair $(\Omega^*(X), d)$ is a cochain complex, i.e.*

$$d^2 = 0.$$

Theorem 2.4. *The exterior derivative is given by the formula*

$$d\omega(x)v_0v_1 \ldots v_k = \sum_{j=0}^{k} (-1)^i (D\omega(x)v_i)v_0, \ldots \hat{v}_i \ldots v_k$$

where the hat indicates that the term under it does not appear, i.e. $\ldots \hat{v}_i \ldots$ means $\ldots v_{i-1}v_{i+1} \ldots$.

Theorem 2.5. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open and $f : X \to Y$ be a smooth map. Then the induced map $f^* : \Omega^k(Y) \to \Omega^k(X)$ commutes with the exterior derivative:*

$$f^*\, d\omega = d\, f^*\omega$$

for $\omega \in \Omega^k(Y)$.

2.6. It is easy to calculate with differential forms. First note that the coordinate functions

$$\mathbb{R}^m \to \mathbb{R} : (x_1, x_2, \dots, x_m) \mapsto x_i$$

restrict to elements $\Omega^0(X)$ whenever $X \subset \mathbb{R}^m$ is open. Then note that the k-fold wedge products

$$d\,x_I := dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $I = (i_1, i_2, \dots, i_k)$ and $1 \le i_1 < i_2 < \cdots < i_k \le m$ give a basis for $\Lambda^k(\mathbb{R}^m)$. Hence any k-form $\omega \in \Omega^k(X)$ has the form

$$\omega = \sum_I a_I(x)\,dx_I.$$

By the axioms for the exterior derivative we have

$$d\omega = \sum_{k=1}^m \sum_I \frac{\partial a_I}{\partial x_k}\,dx_k \wedge dx_I.$$

Any term on the right where $k \in I$ vanishes and the wedge products can be sorted at the expense of changing a few signs. If $f : X \to Y$ and

$$\eta = \sum_I b_J(y)\,dy_J$$

is a k-form on $Y \subset \mathbb{R}^n$ then

$$f^*\eta = \sum_J b_I(f(x))\,df_{j_1} \wedge df_{j_2} \wedge \cdots \wedge df_{j_k}$$

where the sum is over all $J = (j_1, j_2, \dots, j_k)$ with $1 \le j_1 < j_2 < \cdots < j_k \le n$.

Exercise 2.7. Show that if $m = n$ then

$$f^*(dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n) = \det(DF(x))(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n).$$

Exercise 2.8. Let $f : (0,\infty) \times (0, 2\pi) \times (0,\pi) \to \mathbb{R}^3$ be the spherical coordinates map $f(\rho, \theta, \phi) = (x, y, z)$ where

$$x = \rho\cos\phi\cos\theta, \quad x = \rho\cos\phi\sin\theta, \qquad z = \rho\sin\phi.$$

Compute $f^*(dx \wedge dy \wedge dz)$ and $f^*(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$.

Exercise 2.9. Let h be a function defined (on an open subset of) $\mathbb{R}^3$ and $v = (v_1, v_2, v_3)$ be a vector field on $\mathbb{R}^3$. Define the **gradient vector field** ∇h, the **curl** $\nabla \times v$, and the **divergence** $\nabla \cdot v$ as in vector calculus and let

$$\omega = v_1\,dy \wedge dz + v_2\,dz \wedge dx + v_3\,dx \wedge dy, \qquad \lambda = v_1\,dx + v_2\,dy + v_3\,dz.$$

Compare the components of the 1-form dh and the gradient vector field ∇h, of the 2-form $d\lambda$ and the curl $\nabla \times v$, and of the 3-form $d\omega$ and the divergence $\nabla \cdot v$. What is $\iota(v)(dx \wedge dy \wedge dz)$? What is $\iota(v)\lambda$? What is $\iota(v)\omega$?

Theorem 2.10. *The operation which assigns to each vector field $v \in \mathcal{X}(X)$ the linear operator $\ell(v) : \Omega^p(X) \to \Omega^p(X)$ defined by*

$$\ell(v)\omega = \frac{d}{dt}\phi_t^*\omega\bigg|_{t=0}, \qquad \phi_0 = \mathrm{id}, \qquad \frac{d}{dt}\phi_t\bigg|_{t=0} = v$$

is well defined, i.e. independent of the choice of the smooth family of maps $\{\phi_t : X \to X\}_{t\in\mathbb{R}}$ satisfying the two conditions on the right. The operator $\ell(v)$ is called the **Lie derivative** *in the direction v.*

Remark 2.11. The proof shows that $\phi_t(x)$ need not be defined on all of $\mathbb{R} \times X$. Hence taking

$$\phi_t(x) = x + tv(x)$$

gives the correct answer. (For each $x \in X$ point $x + tv(x)$ lies in X for t sufficiently small.)

Theorem 2.12 (Cartan's infinitesimal homotopy formula). *The Lie derivative $\ell(v)$, the exterior derivative d, and interior multiplication $\iota(v)$ are related by the formula*

$$\ell(v) = \iota(v)d + d\iota(v).$$

Proof. Since d raises degree by one and $\iota(v)$ lowers degree by one so both sides preserve degree. The exterior derivative and interior product are skew derivations so their anti commutator (the right hand side of the formula to be proved) is a derivation. The Lie derivative is a derivation as in the proof of the product rule from calculus. Both side commute with d: the left hand side because ϕ_t^* does and the right hand side because $d^2 = 0$. Both sides agree on 0-forms. Hence both sides agree on exact one forms dh. Hence both sides agree on forms of form $h_0\, dh_1 \wedge \cdots \wedge dh_p$. Any form is a finite sum of such forms so the formula holds in general. $\square$

Exercise 2.13. Let

$$\mathrm{O}_3 := \{\Phi \in \mathrm{GL}_3(\mathbb{R}) : \Phi^* = \Phi^{-1}\}$$

denote the group of all linear transformations $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ which preserve distance and

$$\mathrm{SO}_3 := \{\Phi \in \mathrm{SO}_3 : \det(\Phi) = 1\}$$

denote the subgroup of those transformations which preserve orientation as well. Define $\omega \in \Omega^2(\mathbb{R}^3)$ by

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

where x, y, z are the usual coordinates on $\mathbb{R}^3$. Let $\rho := \sqrt{x^2 + y^2 + z^2}$ be the distance from the origin and let $\eta \in \Omega^2(\mathbb{R}^3 \setminus 0)$ be a two form defined in the complement of the origin. Show that

(i) $\Phi^*\omega = \omega$ for all $\Phi \in \mathrm{SO}_3$.

(ii) If $\Phi^*\eta = \eta$ for all $\Phi \in \mathrm{SO}_3$, then $\eta = f(\rho)\omega$ for some smooth $f : (0, \infty) \to \mathbb{R}$.

(iii) If $\Phi^*\eta = \eta$ for all $\Phi \in \mathrm{SO}_3$ and $d\eta = 0$, then $\eta = c\rho^{-3}\omega$ for some $c \in \mathbb{R}$.

3 Calculus on Manifolds

3.1. For a smooth manifold M we denote its tangent bundle by

$$TM := \bigsqcup_{p\in M} T_pM$$

and by

$$\Lambda^k(TM) := \bigsqcup_{p\in M} \Lambda^k(T_pM)$$

the vector bundle whose fiber at a point $p \in M$ is the vector space $\Lambda^k(T_pM)$ of alternating k-multilinear maps on the tangent space at p. For any smooth vector bundle

$$E = \bigsqcup_{p\in M} E_p$$

over M we denote by

$$\Gamma(E) := \{s \in C^\infty(M, E) : s(p) \in E_p,\ \forall p \in M\}$$

the vector space of smooth sections of E. The elements of

$$\mathcal{X}(M) := \Gamma(TM)$$

are called smooth **vector fields** on M and the elements of

$$\Omega^k(M) := \Gamma(\Lambda^k(TM))$$

are called **differential forms** on M. A smooth map $f : M \to N$ induces a linear map $f^* : \Omega^k(N) \to \Omega^k(M)$ called the **pull back** operator via

$$(f^*\eta)(p) := Df(p)^*\eta(f(p))$$

where $T_pf : T_pM \to T_{f(p)}N$ is the **tangent map**:

$$(T_pf)\hat{p} := \frac{d}{dt} f(c(t))\bigg|_{t=0}$$

whenever $c : \mathbb{R} \to M$ satisfies $c(0) = p$ and $\dot{c}(0) = \hat{p} \in T_pM$. When f is a diffeomorphism we define the **push forward** operators $f_* : \Omega^k(M) \to \Omega^k(N)$ and $f_* : \mathcal{X}(M) \to \mathcal{X}(N)$ via

$$f_*\omega = (f^{-1})^*\omega, \qquad (f_*v)(q) = (Tf) \circ v \circ f^{-1}$$

for $\omega \in \Omega^k(M)$, $v \in \mathcal{X}(M)$ and $Tf : TM \to TN$ is defined by $Tf|T_pM := T_pf$.

Remark 3.2. The operation $f \mapsto Tf$ is functorial: $T(f \circ g) = Tf \circ Tg$. If $X \subset \mathbb{R}^m$ is open then then $TX = X \times \mathbb{R}^m$, and if $f : X \to Y$ is a smooth map between open subsets of $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, then

$$Tf(x, \hat{x}) = (f(x), Df(x)\hat{x}).$$

3.3. A **chart** on a manifold M is a diffeomorphism $\phi : U \to X$ where $U \subset M$ and $X \subset \mathbb{R}^m$ are open. When $\psi : V \to Y$ is another chart the diffeomorphism

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is called the **transition map**. A chart might also be called a **local coordinate system** and the transition map might be called the **change of coordinates** map. Each chart ϕ gives rise to a local trivializations of the tangent bundle $TU \subset TM$ and thus a vector field $v \in \mathcal{X}(M)$ determines a vector field $\phi_*(v|U) \in \mathcal{X}(X) = C^\infty(X, \mathbb{R}^m)$ called the **local representative**. Similarly a differential form $\omega \in \Lambda^k(TM)$ determines a local representative $\phi_*\omega|U \in \Omega^k(X) = C^\infty(X, \Lambda^k(\mathbb{R}^m))$. The operations of wedge product, interior product, and exterior derivative all commute with the operators induced by the transition maps and hence determine operations

$$\Omega^{k_1}(M) \times \Omega^{k_2}(M) \to \Omega^{k_1+k_2} : (\omega, \omega_2) \mapsto \omega_1 \wedge \omega_2,$$

$$\mathcal{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M) : (v, \omega) \mapsto \iota(v)\omega,$$

$$\Omega^k(M) \to \Omega^{k+1}(M) : \omega \mapsto d\omega,$$

$$\mathcal{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M) : (v, \omega) \mapsto \ell(v)\omega,$$

which are independent of the choice of coordinates used to define them.

4 Integration

4.1. Let M be a connected smooth manifold of dimension m. Then the bundle $\Lambda^m(TM) \to M$ has a one dimensional fiber. The manifold M is said to **orientable** iff this bundle is trivial, i.e. iff there is a nowhere zero differential m-form on M. Such a form is called a **volume form** on M. Any other volume form is obtained by multiplication by a nowhere zero function. If this function is positive, the two volume forms are said to determine the same **orientation**, otherwise they determine opposite orientations. Thus an orientable manifold has two orientations; an **oriented manifold** is an orientable manifold equipped with one of its two orientations. A chart $\phi : U \to X$ on an oriented manifold is said to be **positively oriented** iff the push forward of some (and hence any) volume form determining the orientation is a positive multiple of $dx_1 \wedge \ldots dx_m$ where $x_1, \ldots, x_m$ are the standard coordinates on $\mathbb{R}^m$. A frame $v_1, v_2, \ldots, v_m \in T_pM$ is said to be **positively oriented** iff the value on some (and hence any) volume form in the orientation is positive. (A **frame** for a vector space is an ordered basis.)

4.2. Let W be a subset of a smooth manifold M. Call W **domain with smooth corners** iff for every point $p \in W$ there is a chart $\phi : U \to X$ on M such that $\phi(U \cap W)$ is a relatively open subset of $X \cap [0, \infty)^m$.

4.3. The **support** of a real valued function (or more generally of any section of a vector bundle) is the closure of the set where it is not zero. A **partition of unity** is a collection of non-negative functions $\{\theta_i\}_{i\in I}$ with **locally finite** supports (meaning that every point has a neighborhood U such that $U\cap\mathrm{supp}(\theta_i)=\emptyset$ for all but at most finitely many i) such that $\sum_{i\in I}\theta_i=1$. (As every point has a neighborhood U such that all but finitely many of the θ_i vanish identically, the meaning of the sum is clear even when the index set I is infinite.) A partition of unity $\{\theta_i\}_{i\in I}$ is said to be **subordinate** to an open cover $\{U_i\}_{i\in I}$ iff

$$\mathrm{supp}(\theta_i)\subset U_i$$

for all $i\in I$. Denote by

$$\Omega^k_c(M):=\{\omega\in\Omega^k(M):\mathrm{supp}(\omega)\text{ is compact}\}$$

the space of forms of compact support.

Theorem 4.4. *Every open cover of a smooth manifold*[1] *admits a subordinate partition of unity.*

Corollary 4.5. *Let $\omega\in\Omega^k_c(M)$ and $\{\phi_i:U_i\to X_i\}_{i\in I}$ be a cover of* $\mathrm{supp}(\omega)$ *by charts. Then ω may be written as a finite sum*

$$\omega=\sum_i\phi_i^*\omega_i$$

where $\omega_i\in\Omega^k_c(X_i)$.

Definition 4.6. Let M be oriented, $W\subset M$ be a domain with smooth corners, and let $\mu\in\Omega^m_c(M)$ have compact support. Then the **integral** of μ over W is defined by

$$\int_W\mu:=\sum_{i\in I}\int_{\phi_i(U_i\cap W)}\mu_i$$

where $\{\phi_i:U_i\to X_i\}_{i\in I}$ is a cover of W by positively oriented charts as in 4.2 and $\mu=\sum_i\phi_i^*\mu_i$ and $\mu_i\in\Omega^m_c(X_i)$ as in Corollary 4.5. (The integration on the right is the Riemann integral.)

Lemma 4.7. *This definition is independent of the cover by positively oriented charts and the decomposition $\mu=\sum_i\phi_i^*\mu_i$ used to define it.*

Remark 4.8. The distinction between the integral of a differential form and the Riemann integral is that the former is an "oriented" integral. For the former interchanging the order of integration reverses the sign of the integrand whereas

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[1] Recall that a smooth manifold is assumed Hausdorff and second countable unless the contrary is explicitly stated.

for the latter the answer is unchanged. Thus if h has compact support

$$\begin{aligned}\int_{\mathbb{R}^2} h(x,y)\,dx\wedge dy &= \int_{y=-\infty}^{\infty}\left(\int_{x=-\infty}^{\infty} h(x,y)\,dx\right)\,dy \\ &= \int_{x=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty} h(x,y)\,dy\right)\,dx \\ &= -\int_{\mathbb{R}^2} h(x,y)\,dy\wedge dx.\end{aligned}$$

Example 4.9. Let $M=\{x\in\mathbb{R}^{m+1}:\sum_i x_i=1\}$ and $W=\Delta^m=M\cap[0,1]^{m+1}$ be the standard m-simplex. Cover W by $m+1$ charts $\phi_i:U_i\to X_i$ where $\phi_i(x)=(x_0,\ldots,\hat{x_i},\ldots x_m)$ discards the ith barycentric coordinate x_i, U_i is the set of $x\in M$ with $x_i>0$, and $X_i\subset\mathbb{R}^m$ is the set of $x\in\mathbb{R}^m$ whose coordinates sum to a number less than 1. Then $\phi_i(U_i\cap W)=X_i\cap[0,\infty)^m$ which proves that the standard m-simplex is a domain with smooth corners. We equip M with the orientation determined by the form $dx_1\wedge\ldots\wedge dx_m$ on M so that ϕ_0 is a positively oriented chart. As $\sum_{i=0}^m dx_i=0$ on M it follows that

$$dx_1\wedge\ldots\wedge dx_m=(-1)^i dx_0\wedge\ldots\widehat{dx_i}\ldots\wedge dx_m$$

on M so that ϕ_i is a positively oriented chart for i even and a negatively oriented chart for i odd.

4.10. Call W **domain with smooth boundary** iff for every point $p\in W$ there is a chart $\phi:U\to X$ on M such that $\phi(U\cap W)$ is a relatively open subset of $X\cap([0,\infty)\times\mathbb{R}^{m-1})$. The **boundary** of such W is defined by the condition that

$$\partial W=\phi^{-1}\bigl(X\cap(0\times\mathbb{R}^{m-1})\bigr)$$

for every such chart. (Note: if $p\in\phi^{-1}\bigl(X\cap(0\times\mathbb{R}^{m-1})\bigr)$ for one such chart, then $p\in\phi^{-1}\bigl(X\cap(0\times\mathbb{R}^{m-1})\bigr)$ for every such chart with $p\in U$.) Thus ∂W is a submanifold of M of dimension $m-1$. This boundary inherits an orientation from an orientation of M as follows. For $p\in\partial W$ call a vector $v\in T_pM$ **outward pointing** iff $T_pM=\mathbb{R}v\oplus T_p\partial W$ and $v=\dot{c}(0)$ where $c:\mathbb{R}\to M$ is a smooth curve satisfying $c(0)=p$ and $c(t)\in W$ for $t\le 0$. Then a frame $v_1,\ldots,v_{m-1}\in T_pW$ is positively oriented iff $v,v_1,\ldots,v_{m-1}$ for some (and hence every) outward pointing vector v. A domain with smooth boundary is in particular a domain with smooth corners so the integral of a differential form over it is defined as in 4.6.

Theorem 4.11 (Stokes' Theorem). *Let W be a domain with smooth boundary in an oriented manifold M of dimension m and let $\omega\in\Omega_c^{m-1}(M)$. Then*

$$\int_W d\omega=\int_{\partial W}\omega.$$

Proof. It is enough to prove this in the special case $M=\mathbb{R}^m$, $W=0\times\mathbb{R}^{m-1}$, and $\omega=h(x)\,dx_1\wedge\cdots\widehat{dx_k}\cdots\wedge dx_m$ so $d\omega=(-1)^{k-1}(\partial h/\partial x_k)\,dx_1\wedge\ldots\wedge dx_m$.

In this special case, the vector $v = (-1, 0, \dots, 0)$ is outward pointing along the boundary. In case $k > 1$ the left side of Stokes' Theorem vanishes by Fubini's Theorem and the Fundamental Theorem of Calculus and the right side vanishes because $dx_1 = 0$ on the boundary $x_1 = 0$. In case $k = 1$ we have

$$\int_W d\omega = \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial h}{\partial x_1}\, dx_1 \right) dx_2 \dots dx_m$$

by Fubini's Theorem so by the Fundamental Theorem of Calculus we have

$$\int_W d\omega = -\int_{\mathbb{R}^{m-1}} h(0, x_2, \dots, x_m)\, dx_2 \dots dx_m$$

But

$$\int_{\mathbb{R}^{m-1}} h(0, x_2, \dots, x_m)\, dx_2 \dots dx_m = -\int_{\partial W} h\, dx_2 \wedge \dots \wedge dx_m = \int_{\partial W} \omega$$

as $dx_2 \wedge \dots \wedge dx_m$ determines the opposite orientation of ∂W. □

4.12. A **smooth singular n-simplex** in a manifold M is a smooth map $\sigma : \Delta^n \to M$; a **smooth singular n-chain** in M is an element of the real vector space generated by the smooth singular n-simplices, i.e. a finite formal sum

$$c = \sum_{i=1}^{n} c_i \sigma_i$$

where $c_i \in \mathbb{R}$ and $\sigma_i : \Delta^n \to M$ is smooth. Denote by $S_n(M)$ the vector space of all smooth singular n-chains in M and define a boundary operator

$$\partial : S_n(M) \to S_{n-1}(M)$$

by

$$\partial\sigma = \sum_{k=0}^{n} (-1)^k \sigma \circ \iota_k$$

where $\iota_k : \Delta^{n-1} \to \Delta^n$ is the kth **face**

$$\iota_n(y_1, \dots, y_n) = (y_1, \dots, y_{k-1}, 0, y_{k+1}, \dots, y_n).$$

The standard n-simplex Δ^n has a **standard orientation** as explained in Example 4.9 so we may define the integral of a differential form $\omega \in \Omega^n(M)$ over the smooth singular n-chain c by

$$\int_c \omega := \sum_{i=1}^{n} c_i \int_{\Delta^n} \sigma_i^* \omega.$$

Theorem 4.13 (Stokes' Theorem for Chains). *For $\omega \in \Omega^{n-1}(M)$ and $c \in S_n(M)$ we have*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof. The proof is somewhat trickier than the proof in [4] as we are using simplicial chains rather than cubical chains. By linearity it is enough to treat the special case $c = \sigma$ and as $\sigma^* d = d\sigma^*$ we may replace ω by $\sigma^*\omega$ and assume that σ is the identity. Thus we must prove

$$\int_{\Delta^n} d\omega = \sum_{k=0}^{n} (-1)^k \int_{\Delta^{n-1}} \iota_k^* \omega.$$

Denote by V the common tangent space to Δ^n at each of its points and by W the common tangent space to Δ^{n-1}. Thus a vector $v \in \mathbb{R}^{n+1}$ lies in V iff its coordinates sum to zero. Note that $\iota_k : W \to V$. Orient the jth face as usual, i.e. a frame $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ for $\iota_k(W)$ is positively oriented iff the frame $(\nu_k, \varepsilon_1, \ldots, \varepsilon_{n-1})$ is positively oriented in V where ν_k points out of Δ^n along the kth face.

Claim. *The face map $\iota_k : \Delta^{n-1} \to \Delta^n$ preserves orientation if k is even and reverses orientation if k is odd.* To see this introduce the forms

$$\lambda_k = dx_1 \wedge \ldots \widehat{dx_k} \ldots \wedge dx_n$$

for $k = 1, \ldots, n$ and let $\lambda_0 = \lambda_1$. Then $\iota_k^* \lambda_k = dy_2 \wedge \ldots \wedge dy_n$ where $y_1, \ldots, y_n$ are the barycentric coordinates on Δ^{n-1} The form dx_k is negative on an outward pointing vector on the kth face (i.e. where $x_k = 0$) and

$$dx_1 \wedge \cdots \wedge dx_n = (-1)^{k-1} dx_k \wedge \lambda_k.$$

(For $k = 0$ this is because $dx_0 + \cdots + dx_n = 0$.) It follows that λ_k induces the orientation on the kth face if and only if k is even. This proves the claim.

To complete the proof write $\omega = \sum_{j=0}^{n} \omega_j$ where ω_j is supported in the set $x_j > 0$ and choose coordinates taking values in $[0, \infty)^n$ by discarding x_j (see Definition 4.6). Then argue as in Theorem 4.11 for each ω_j. In these coordinates the outward pointing normal on the kth face is the negative of the kth standard basis vector in $\mathbb{R}^n$ so the integral over this face appears with the correct sign by the claim. □

Remark 4.14. It is not hard to prove a version of Stokes' Theorem for domains with smooth corners but this would require defining manifolds with corners not just domains with corners. A careful exposition would be a little unpleasant. Note for example, that it would be reasonable to view a convex polyhedron in $\mathbb{R}^3$ as a manifold with corners, but a convex polyhedron in $\mathbb{R}^3$ could have more than three faces intersecting at a vertex and would hence not be locally diffeomorphic to $[0, \infty)^3$.

5 De Rham cohomology

5.1. Recall that **chain complex** is a sequence C_* of Abelian groups and group homomorphisms $\partial : C_k \to C_{k-1}$ called **boundary operators** such that $\partial^2 = 0$.

A **real chain complex** is defined the same way except that we read *real vector space* for *Abelian group* and *real linear map* for *homomorphism.* The notions of **chain map**, **chain homotopy**, **chain equivalence**, are similarly defined for real chain complexes. The **homology**

$$H_k(C) := \frac{\mathrm{kernel}\bigl(\partial : C_k \to C_{k-1}\bigr)}{\mathrm{image}\bigl(\partial : C_{k+1} \to C_k\bigr)}.$$

is now a real vector space. In the sequel we only consider real chain complexes so we drop the modifier *real.* We only consider complexes where $C_k = 0$ for $k < 0$.

5.2. A **cochain complex** is defined in the same way as a chain complex except that the boundary operator increases the index. It is customary use superscripts for cochain complexes. The **cohomology** of a cochain complex (C^*, δ) is the quotient space

$$H^k(C) := \frac{\mathrm{kernel}\bigl(\delta : C^k \to C^{k+1}\bigr)}{\mathrm{image}\bigl(\delta : C^{k-1} \to C^k\bigr)}.$$

The theory of cochain complexes is exactly the same as the theory of chain complexes, only the indexing is different. It is customary to call elements of a chain complex $C_*, \partial)$ **chains**, elements in the kernel of ∂ **cycles** and elements in the image of ∂ **boundaries**. For cochain complexes we usually speak of **cochains**, **cocycles**, and **coboundaries**.

5.3. Every chain complex (C_*, ∂) determines a cochain complex (C^*, δ) called the **dual complex** defined by

$$C^k = C_k^*, \qquad \delta = \partial^*$$

where V^* denotes the dual vector space of all real linear maps from the vector space V to the real numbers and $\phi^*(\gamma) = \gamma \circ \phi$ for a linear map $\phi : V \to W$ and an element $\gamma \in W^*$. Since $(\phi \circ \partial)^* = \partial^* \circ \phi^*$ and $(\partial \circ \phi)^* = \phi^* \circ \partial^*$, the dual of a chain map is a cochain map. Similarly chain homotopies dualize to chain homotopies. For any chain complex (C, ∂) there is a unique linear map $\kappa : H^k(C) \to H_k(C)^*$ called the **Kronecker product** such that

$$\kappa([\gamma])([c]) = \gamma(c)$$

whenever $\gamma \in C^k$, $\partial^*\gamma = 0$, $c \in C_k$, $\partial c = 0$; here $[\gamma] \in H^k(C)$ denotes the cohomology class of the cocycle γ and $[c] \in C_k$ denotes the homology class of the cycle c.

Theorem 5.4. *The Kronecker product is an isomorphism of vector spaces.*

Proof. Write $C_k = A_k \oplus H_k \oplus B_k$ where ∂ maps A_k isomorphically onto B_k and $\partial = 0$ on H_k and B_k. □

Definition 5.5. The kth **de Rham cohomology** $H^k_{DR}(M)$ of the smooth manifold M is the kth cohomology group of the **de Rham complex**

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & \Omega^{k-1}k+1(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \cdots . \end{tikzcd}$$

A smooth map $f : M \to N$ induces a cochain map $f^* : \Omega^k(N) \to \Omega^k(M)$ and hence also a map $H^k(N) \to H^k(M)$ again denoted by f^*.

Theorem 5.6. *Smoothly homotopic maps induce chain homotopic maps on differential forms and hence the same map in de Rham cohomology.*

Proof. Let $\{f_t : M \to N\}_{t\in[0,1]}$ be a smooth homotopy, i.e. the map $F : M \times [0,1] \to N$ given by $F(p,t) = f_t(p)$ is smooth. Let $v \in \mathcal{X}(M \times [0,1])$ be defined by $v(p,t) = (0,1) \in T_pM \times \mathbb{R}$. Define a chain homotopy operator $K : \Omega^*(N) \to \Omega^{*-1}(M)$ by

$$K(\omega) = \int_{t=0}^{1} \iota(v) F^*\omega \, dt.$$

By Cartan's infinitesimal homotopy formula 2.12

$$f_1^*\omega = f_0^*\omega = \int_0^1 \frac{df_t^*\omega}{dt}\, dt = dK(\omega) + K(d\omega)$$

so f_1 and f_0 are chain homotopic. $\square$

Corollary 5.7 (Poincaré Lemma). *If M is smoothly contractible, then*

$$H^k_{DR}(M) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k > 0. \end{cases}$$

Theorem 5.8 (Mayer Vietoris). *Let U and V be open subsets of a manifold M. Then the sequence*

$$\begin{tikzcd} 0 \arrow[r] & \Omega^k(U \cup V) \arrow[r, "\alpha"] & \Omega^k(U) \oplus \Omega^k(V) \arrow[r, "\beta"] & \Omega^k(U \cap V) \arrow[r] & 0 \end{tikzcd}$$

is exact where $\alpha(\omega) = (\omega|U) \oplus (\omega|V)$ and $\beta(\omega_1 \oplus \omega_2) = (\omega_1|U\cap V) - (\omega_2|U \cap V)$. Hence there is a long exact sequence

$$\cdots H^k_{DR}(U \cup V) \to H^k_{DR}(U) \oplus H^k_{DR}(V) \to H^k_{DR}(U \cap V) \to H^{k+1}_{DR}(U \cup V) \cdots .$$

The long exact sequence is natural with respect to smooth maps from $U \cup V$ to $U' \cup V'$ which send U to U' and V to V'.

Definition 5.9. The kth **smooth singular homology** $H_k(M)$ of the smooth manifold M is the kth homology group of the **smooth singular chain complex**

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & S_{k+1}(M) \arrow[r, "\partial"] & S_k(M) \arrow[r, "\partial"] & S_{k-1}(M) \arrow[r, "\partial"] & \cdots . \end{tikzcd}$$

A smooth map $f : M \to N$ induces a chain map $f_* : S_k(M) \to S_k(N)$ and hence also a map $H_k(M) \to H_k(N)$ again denoted by f^*. The map f_* is uniquely determined by $f_*\sigma = f \circ \sigma$ for each smooth singular simplex $\sigma : \Delta^n \to M$.

Theorem 5.10. *Smoothly homotopic maps induce chain homotopic maps on smooth singular chains and hence the same map in smooth singular homology.*

Corollary 5.11 (Poincaré Lemma). *If M is smoothly contractible, then*

$$H_k(M) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k > 0. \end{cases}$$

Lemma 5.12. *Let U and V be open subsets of a manifold M and $S_k(U, V) \subset S_k(U \cup V)$ denote subcomplex of all chains $c = \sum_{i=1}^n c_i\sigma_i$ where $\sigma_i \in S_k(U) \cup S_k(V)$. Then the inclusion $S_k(U, V) \to S_k(U \cup V)$ is a chain homotopy equivalence.*

Theorem 5.13 (Mayer Vietoris). *Let U and V be open subsets of a manifold M. Then the sequence*

$$\begin{tikzcd} 0 \arrow[r] & S_k(U \cap V) \arrow[r, "i"] & S_k(U) \oplus S_k(V) \arrow[r, "j"] & S_k(U, V) \arrow[r] & 0 \end{tikzcd}$$

is exact where $i(\sigma) = (\sigma) \oplus (-\sigma)$ and $j(\sigma_1 \oplus \sigma_2) = \sigma_1 + \sigma_2$. This induces a long exact sequence

$$\cdots H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(U \cup V) \to H_{k-1}(U \cap V) \cdots .$$

The long exact sequence is natural with respect to smooth maps from $U \cup V$ to $U' \cup V'$ which send U to U' and V to V'.

Definition 5.14. The kth **smooth singular cohomology** $H^k(M)$ of the smooth manifold M is the kth cohomology group of the **smooth singular cochain complex**

$$\begin{tikzcd} \cdots \arrow[r, "\partial^*"] & S^{k-1}(M) \arrow[r, "\partial^*"] & S^k(M) \arrow[r, "\partial^*"] & S^{k+1}(M) \arrow[r, "\partial^*"] & \cdots \end{tikzcd}$$

which is by definition the dual complex to the smooth singular chain complex. A smooth map $f : M \to N$ induces a linear map $f^* : S^k(N) \to S^k(M)$ which is dual to $f_* : S_k(M) \to S_k(N)$ and hence a linear map $H^k(N) \to H^k(M)$ also denoted by f^*.

Theorem 5.15. *Smoothly homotopic maps induce chain homotopic maps on smooth singular cochains and hence the same map in smooth singular cohomology.*

Proof. The dual of a chain homotopy is a chain homotopy. ☐

Corollary 5.16 (Poincaré Lemma). *If M is smoothly contractible, then*

$$H^k(M) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k > 0. \end{cases}$$

Lemma 5.17. *Let U and V be open subsets of a manifold M and $S^k(U,V) := S_k(U,V)^*$. Then the restriction $S^k(U \cup V) \to S^k(U,V)$ is a chain homotopy equivalence.*

Proof. The dual of a chain homotopy is a chain homotopy. □

Theorem 5.18 (Mayer Vietoris). *Let U and V be open subsets of a manifold M. Then the sequence*

$$\begin{tikzcd} 0 \arrow[r] & S^k(U,V) \arrow[r, "j^*"] & S^k(U) \oplus S^k(V) \arrow[r, "i^*"] & S^k(U \cap V) \arrow[r] & 0 \end{tikzcd}$$

obtained by dualizing the exact sequence of Theorem 5.13 is exact. This induces a long exact sequence so there is a long exact sequence

$$\cdots H^k(U \cup V) \to H^k(U) \oplus H_k(V) \to H^k(U \cap V) \to H^{k+1}(U \cup V) \cdots .$$

The long exact sequence is natural with respect to smooth maps from $U \cup V$ to $U' \cup V'$ which send U to U' and V to V'.

Proof. The dual of a short exact sequence of vector spaces is exact and dualizing a commutative diagram of chain maps between short exact sequences gives another such. (Note: This argument depends on the fact that the coefficients are from a field.) □

Theorem 5.19 (De Rham). *Let M be smooth manifold M. Then the linear map $I : \Omega^k(M) \to S^k(M)$ defined by*

$$I(\omega)(c) := \int_c \omega.$$

is a chain map and induces an isomorphism $H^*_{DR}(M) \to H^*(M)$ between de Rham cohomology and smooth singular cohomology.

5.20. A **good cover** of an m-dimensional manifold M is an open cover $\{U_i\}_{i \in A}$ (here A is an index set) such that for $k \leq m$ every $(k+1)$-fold intersection

$$U_\alpha := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$$

is diffeomorphic to $\mathbb{R}^m$ and for $k > m$ every $(k+1)$-fold intersection of distinct elements of the cover is empty. Given any cover we can find a subordinate good cover; for example we might triangulate M, subdivide so that each star is a subset of an element of the cover, and take the open stars of vertices. (Some further argument would be needed to show that the stars are diffeomorphic to $\mathbb{R}^m$.) One could also construct good covers using a Riemannian metric and geodesically convex open sets. (A finite intersection of geodesically convex open sets is geodesically convex.)

We shall always assume our manifolds have finite good covers. This is not always true (e.g. $M = \mathbb{R}^2 \setminus \mathbb{Z}^2$) but this class of manifolds includes compact manifolds. We shall never need the requirement that $(m+2)$-fold intersections are empty and often will require only that the slightly weaker condition that the U_α are smoothly contractible. The appendix to [3] shows how to extend our arguments to general (paracompact) smooth manifolds.

Exercise 5.21. Show that an open star shaped region in $\mathbb{R}^m$ with compact closure is diffeomorphic to $\mathbb{R}^m$. Hint: There is a nonnegative function h such that $h^{-1}((0,\infty))$ is the open set. The definition of star shaped is that each ray emanating from the origin intersects the open set in an open interval. The idea is to map this interval diffeomorphically onto $(0,\infty)$ and use polar coordinates to define the diffeomorphism. It need not be true that the endpoint of this interval is a continuous function of the ray, but the idea works anyway as follows. Let $\{\phi^t : \mathbb{R}^m \to \mathbb{R}^m\}$ be the flow of the vector field $v(x) := h(x)x$. Then $\phi^t(x) = x$ if $h(x) = 0$. Let $f(ru) = \phi^r(u)$ for $u \in \mathbb{S}^{m-1}$. Modify f so that this map is identity near the origin. Is the hypothesis that the closure be compact needed?

Remark 5.22. The result of Exercise 5.21 applies to intersections of open stars of vertices in simplicial complexes, for example, the interior of Δ^m.

6 Compactly supported de Rham cohomology

Definition 6.1. The kth **compactly supported de Rham cohomology** $H_c^k(M)$ of the smooth manifold M is the kth homology group of the **compactly supported de Rham complex**

$$\cdots \xrightarrow{d} \Omega_c^{k-1}(M) \xrightarrow{d} \Omega_c^k(M) \xrightarrow{d} \Omega_c^{k+1}(M) \xrightarrow{d} \cdots .$$

A proper smooth map $f : M \to N$ induces a chain map $f^* : \Omega_c^k(N) \to \Omega_c^k(M)$ and hence also a map $H_c^k(N) \to H_c^k(M)$ again denoted by f^*.

Remark 6.2. When M is compact $\Omega_c^k(M) = \Omega^k(M)$ so $H_c^k(M) = H^k(M)$.

Theorem 6.3. *Let $\{f_t : M \to N\}_{t\in\mathbb{R}}$ be a* **proper homotopy***, i.e. the map*

$$M \times \mathbb{R} \to N \times \mathbb{R} : (p,t) \mapsto (f_t(p), t)$$

is proper. Then the induced map $f_t^* : H_c^k(N) \to H_c^k(M)$ is independent of t.

Theorem 6.4 (Poincaré Lemma). *The compactly supported de Rham cohomology of Euclidean space is given by*

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \textit{for } k = m \\ 0 & \textit{otherwise.} \end{cases}$$

The integral

$$\Omega_c(\mathbb{R}^m) \to \mathbb{R} : \omega \mapsto \int_{\mathbb{R}^m} \omega$$

induces an isomorphism $H_c^m(\mathbb{R}^m) \to \mathbb{R}$.

Proof. We treat only the case $m = 1$; the general case follows from Theorem 8.9 below. ☐

Theorem 6.5 (Mayer Vietoris). *Let* U *and* V *be open subsets of a manifold* M. *Then the sequence*

$$\begin{tikzcd} 0 \arrow[r] & \Omega_c^k(U \cap V) \arrow[r, "i"] & \Omega_c^k(U) \oplus \Omega_c^k(V) \arrow[r, "j"] & \Omega_c^k(U \cup V) \arrow[r] & 0 \end{tikzcd}$$

is exact where i *is the inclusion (extension by zero) on each factor and* j *is extension by zero followed by subtraction. This induces a long exact sequence*

$$\cdots H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_x^k(U \cup V) \to H^{k+1}(U \cap V) \cdots .$$

The long exact sequence is natural with respect to smooth proper maps from $U \cup V$ *to* $U' \cup V'$ *which send* U *to* U' *and* V *to* V'.

Theorem 6.6 (Poincaré duality). *For a smooth manifold oriented manifold* M *the linear map* $W : \Omega^k(M) \to \Omega_c^{m-k}(M)^*$ *defined by*

$$W(\omega)(\eta) := \int_M \omega \wedge \eta.$$

is a chain map and induces an isomorphism between de Rham cohomology and compactly supported de Rham cohomology.

Remark 6.7. Note that $C_k := \Omega_c^{m-k}(M)$ is a homology theory in the sense that $d : C_k \to C_{k-1}$. Hence $C^k := \Omega_c^{m-k}(M)^*$ is a cohomology theory.

Corollary 6.8. *Let* S *be a closed oriented submanifold of codimension* r *of an oriented smooth manifold* M *of dimension* m *and* $i : S \to M$ *be the inclusion. Then there is a closed form* $\eta \in \Omega^r(M)$ *such that*

$$\int_S i^*\omega = \int_M \omega \wedge \eta$$

for every closed form $\omega \in \Omega_c^{m-r}(M)$. *The cohomology class* $\eta_S \in H_{DR}^r(M)$ *of* η *is independent of the choice of* η *and is called the* **Poincaré dual of the submanifold** S.

Proof. The linear functional $\Omega_c^{m-r}(M) \to \mathbb{R} : \omega \mapsto \int_S i^*\omega$ (well) defines an element of $H_c^{m-r}(M)^* \simeq H_{DR}^r(M)$. ☐

7 Bundles

7.1. Let F be a manifold. A smooth map $\pi : E \to M$ is called **locally trivial** with fiber F iff for every point $p_0 \in M$ there is a neighborhood U and a diffeomorphism $\Phi : U \times F \to \pi^{-1}(U)$ such that $\pi \circ \Phi(p, v) = p$ for $p \in U$ and $v \in F$. Such a diffeomorphism U is called a **local trivialization**. Another local trivialization $\Psi : V \times F \to \pi^{-1}(V)$ determines a **transition map** $\Psi \circ \Phi^{-1} : (U \cap V) \times F \to \pi^{-1}(U \cap V)$. The transition map has the form

$$\Psi \circ \Phi^{-1}(p, w) = (p, g_p(w))$$

where $g_p : F \to F$ is a diffeomorphism for each $p \in U \cap V$.

7.2. Now let $G \subset \mathrm{diff}(F)$ be a subgroup of the diffeomorphism group of the fiber F. Two local trivializations as in 7.1 are called G**-compatible** iff $g_p \in G$ for all $p \in U \cap V$. A **fiber bundle** with **fiber** F, **structure group** G, **base space** M, and **total space** E is a locally trivial map π equipped with a maximal **atlas** of local trivializations, any two of which are G-compatible, and whose domains cover E. Such a maximal atlas is called a G**-structure** on the locally trivial map. *Maximal* means that any local trivialization which is G-compatible with every local trivialization in the atlas is itself a member of the atlas. As was the case for manifolds, every atlas extends uniquely to a maximal atlas. The **trivial bundle** is the map $\pi : M \times F \to M$ where π is projection on the first factor. A **section** of a fiber bundle is a right inverse to the **projection** π.

Remark 7.3. The definitions make precise the following idea. A fiber bundle is a function which assigns to each point p in a manifold M another manifold

$$E_p := \pi^{-1}(p)$$

called the **fiber** over p and the fiber E_p depends smoothly on p. The fiber of the trivial bundle is constant. A section assigns to each $p \in M$ a point of the fiber over p. A G-structure specifies for each p and equivalence class of diffeomorphisms from the **standard fiber** F to the given fiber E_p called **frames**; two frames are equivalent if on is obtained from the other by composition on the left with an element of $G \subset \mathrm{diff}(F)$.

Definition 7.4. A **vector bundle** is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $\mathrm{GL}_n(\mathbb{R})$, the group of all linear automorphisms of $\mathbb{R}^n$. For a vector bundle each fiber is a vector space since the operations of vector addition and scalar multiplications are independent of the choice of local trivialization used to define them. A frame in a fiber E_p is a linear isomorphism $\mathbb{R}^n \to E_p$; this determines and is determined by the image of the standard basis for $\mathbb{R}^n$ so a frame in E_p may be viewed as an ordered basis for the vector space E_p.

Example 7.5. Each chart $\phi : U \to X$ on a manifold M determines a local trivialization of the tangent bundle $TM \to M$ via the formula

$$\Phi(p, w) = (p, \dot{c}(0)), \qquad c(t) = \phi^{-1}(\phi(p) + tw)$$

for $w \in \mathbb{R}^m$ and $t \in \mathbb{R}$. The transition map between two such local trivializations is given by

$$g_p(w) = D(\psi \circ \phi^{-1})(\phi(p))w.$$

This map is linear in w for each p so the maximal atlas determined by these charts is a vector bundle. A section of the tangent bundle is the same as a **vector field** on M.

Example 7.6. A vector bundle $E \to M$ and integer $k \geq 0$ determine a new vector bundle $\Lambda^k(M, E) \to M$ with fiber

$$\Lambda^k(M, E)_p := \Lambda^k(T_pM, E_p)$$

over $p \in M$. The space of smooth sections of this bundle is denoted by $\Omega^k(M, E)$ and its elements are called **differential forms with values in** E.

7.7. Suppose the fiber F is oriented. A **fiber orientation** is an atlas of local trivializations such that for each transition map $\Psi \circ \Phi^{-1}$ as in 7.1 between elements of the atlas and for each $p \in U \cap V$ the diffeomorphism $g_p : F \to F$ is orientation preserving. A fiber orientation is the same as a G-structure where $G \subset \mathrm{diff}(F)$ is the subgroup of orientation preserving diffeomorphisms, but usually the terminology "G-structure" is only used when the subgroup G is a finite dimensional Lie group.

7.8. Given a bundle $\pi : E \to M$ and a smooth map $f : N \to M$ the **pull back bundle** $f^*\pi : f^*E \to N$ has fiber over $q \in N$ given by

$$(f^*E)_q = E_{f(q)}.$$

Each local trivialization $\Phi : U \times F \to \pi^{-1}(U)$ determines a local trivialization $f^*\Phi : f^{-1}(U) \times F \to (f^*\pi)^{-1}(V)$ via the formula

$$(f^*\Phi)(q, v) = \Phi(f(q), v).$$

In the commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"] \\
N \arrow[r, "f"] & M
\end{tikzcd}$$

the map on the top is just $(q, v) \mapsto (f(q), v)$ and is thus an isomorphism on each fiber.

7.9. The pull back of a G-structure is a G-structure; in particular, the pull back of a vector bundle is a vector bundle, and the pull back of a fiber orientation is a fiber orientation. When N is an open subset U of the manifold M and f is the inclusion the pull back can be viewed as restriction and we use the notation

$$E|U := \pi^{-1}(U)$$

for the total space of the restricted bundle.

Theorem 7.10 (Leray Hirsch). *Let $\pi : E \to M$ be a fiber bundle and assume that M is connected. Suppose that $\eta_1, \dots, \eta_n \in \Omega^*(E)$ are such that for some (and hence every) $p \in M$ the restrictions $\eta_1|E_p, \dots, \eta_n|E_p$ represent a basis for $H^*_{DR}(E_p)$. Let $\mathbb{R}\{\eta_1, \dots, \eta_n\}$ denote the span of these forms. Then the map*

$$\Omega^*(M) \otimes \mathbb{R}\{\eta_1, \dots, \eta_n\} \to \Omega^*(E) : \xi \otimes \eta_i \mapsto \pi^*\xi \wedge \eta_i$$

induces an isomorphism $H^*_{DR}(M) \otimes H^*_{DR}(F) \to H^*_{DR}(E)$.

Proof. First we show “and hence every”, i.e. that if $\eta_1|E_p, \dots, \eta_n|E_p$ represent a basis for $H^*_{DR}(E_p)$, then $\eta_1|E_q, \dots, \eta_n|E_q$ represent a basis for $H^*_{DR}(E_q)$. It is not hard to construct smooth isotopies $\{\phi_t : M \to M\}_{t\in[0,1]}$ and $\{\Phi_t : E \to E\}_{t\in[0,1]}$ satisfying $\phi_0 = \mathrm{id}_M$, $\Phi_0 = \mathrm{id}_E$, and $\pi \circ \Phi_t = \phi_t$. (An **isotopy** is a smooth homotopy $\{f_t : M \to N\}_{t\in[0,1]}$ such that each f_t is a diffeomorphism.) This gives a commutative diagram

$$\begin{tikzcd}
E_p \arrow[r] \arrow[d] & E \arrow[d] \\
E_q \arrow[r] & E
\end{tikzcd}$$

where the horizontal arrows are inclusions, the vertical arrow on the right is Φ_1 and the vertical arrow on the left is $\Phi_1|E_p$. The vertical arrows are diffeomorphisms. By hypothesis the top horizontal map induces an isomorphism in cohomology. The vertical map on the right induces the identity on $H^*_{DR}(E)$ as it is isotopic to the identity and hence preserves the cohomology classes of $\eta_1, \dots, \eta_n$. The vertical map on left induces an isomorphism in cohomology as it is a diffeomorphism. It follows immediately that the bottom horizontal map induces an isomorphism in cohomology and hence pulls the cohomology classes of $\eta_1, \dots, \eta_n$ back to representatives of a basis in $H^*_{DR}(E_q)$.

The map $\xi\otimes\eta_i \mapsto \pi^*\xi\wedge\eta_i$ is a chain map, i.e. it sends $(d\xi)\otimes\eta_{)}$ to $\pi^*(d\xi)\wedge\eta_i$. BLAH ☐

Corollary 7.11 (Künneth Formula). *The map*

$$\Omega^*(M) \otimes \Omega^*(F) \to \Omega^*(M \times F) : \xi \otimes \eta \mapsto \pi_M^*\xi \wedge \pi_F^*\eta$$

induces isomorphisms

$$\bigoplus_{i+j=k} H^i_{DR}(M) \otimes H^j_{DR}(F) \cong H^k_{DR}(M \times F).$$

8 Compact vertical cohomology

8.1. A form $\omega \in \Omega^n(E)$ where E is the total space of a fiber bundle $\pi : E \to M$ is said to have **compact vertical support** iff for every point $p \in M$ there is a local trivialization $\Phi : U \times F \to \pi^{-1}(U)$ and a compact subset $K \subset F$ such that $p \in U$ and $\mathrm{supp}(\omega) \cap \pi^{-1}(U) \subset \Phi(U \times K)$. The space of all n-forms on E of compact vertical support is denoted by $\Omega^n_{cv}(E)$.

Definition 8.2. For a fiber bundle $\pi : E \to M$ the kth **compact vertical cohomology** $H^k_{cv}(E)$ is the kth homology group of the complex

$$\cdots \xrightarrow{d} \Omega^{k-1}_{cv}(E) \xrightarrow{d} \Omega^k_{cv}(E) \xrightarrow{d} \Omega^{k+1}_{cv}(E) \xrightarrow{d} \cdots .$$

A smooth map $f : N \to M$ induces a chain map $f^* : \Omega^k_{cv}(E) \to \Omega^k_{cv}(f^*E)$ and hence also a map $H^k_{cv}(E) \to H^k_{cv}(f^*E)$ again denoted by f^*.

Remark 8.3. When the fiber F is compact $\Omega^k_{cv}(E) = \Omega^k(E)$ $H^k_{cv}(E) = H^k(E)$.

8.4. Let $\pi : E \to M$ be a fiber bundle and r be the dimension of the fiber F. The function which assigns to each $p \in M$ the infinite dimensional vector space $\Omega * n(E_p)$ can be viewed as a vector bundle over M. When $n = r$ a differential form $\omega \in \Omega^{k+r}(E)$ on the total space E determines a section $R\omega$ of this infinite dimensional bundle via the formula

$$(R\omega)_p(v_1, \ldots, v_k)_q(u_1, \ldots, u_r) = \omega_q(v'_1, \ldots, v'_k, u_1, \ldots, u_r)$$

for $q \in E_p$, $u_1, \ldots, u_r \in T_qE_p$, $v_1, \ldots, v_k \in T_pM$, and $v'_1, \ldots, v'_k \in T_qE$ satisfying $d\pi(q)v'_i = v_i$ for $i = 1, \ldots, k$. (Because $n = r$, the definition is independent of the choice of such $v'_1, \ldots, v'_k$.)

Definition 8.5. Let $\pi : E \to M$ be a fiber oriented bundle and r be the dimension of the fiber F. The operation $\pi_* : \Omega^{k+r}_{cv}(E) \to \Omega(M)$ defined by

$$(\pi_*\omega)_p(v_1, \ldots, v_k) = \int_{E_p} (R\omega)_p(v_1, \ldots, v_k)$$

with R as in 8.4 is called **integration over the fiber**.

Theorem 8.6 (Projection Formula). *Let $\pi : E \to M$ be a fiber oriented bundle. Then*

$$\pi_*(\pi^*\eta \wedge \omega) = \eta \wedge \pi_*\omega$$

for $\eta \in \Omega^*(M)$ and $\omega \in \Omega^*_{cv}(E)$. Moreover

$$\int_E \pi^*\eta \wedge \omega = \int_M \eta \wedge \pi_*\omega$$

if the degrees of η and ω add up to the dimension of the manifold E.

Example 8.7. The idea of compact vertical support is that restriction to the fiber produces a form of compact support so that fiber integration can be defined. However the precise definition requires some uniformity as illustrated by the following. Let $f \in \Omega^0_c(\mathbb{R})$ satisfy $\int f(t)\,dt = 1$ and $f(0) = 0$. Take $M = \mathbb{R}$, $E = M \times \mathbb{R}$, and $\omega \in \Omega^1(E)$ be given by $\omega = pf(pt)\,dt$ for $(p, t) \in M \times \mathbb{R} = E$ Then $\int_{\mathbb{R}} pf(pt)\,dt = 1$ for $p \neq 0$ but the integral vanishes for $p = 0$. The fiber integral is not continuous. The restriction of ω to each fiber has compact support but $\omega \notin \Omega^1_{cv}(E)$.

Theorem 8.8. *For a fiber oriented bundle $\pi : E \to M$ integration over the fiber is a chain map.*

Theorem 8.9 (Poincaré Lemma). *For a trivial bundle $E = M \times \mathbb{R}^r$ integration over the fiber induces isomorphisms*

$$H^*_{cv}(E) \to H^{*-r}_{DR}(M) \qquad \text{and} \qquad H^*_c(E) \to H^{*-r}_c(M).$$

Theorem 8.10 (Compact Vertical Mayer Vietoris). *Let $\pi : E \to M$ be a fiber oriented bundle and U and V be open subsets of M. Then the sequence*

$$0 \to \Omega^k_{cv}(E|U \cup V) \xrightarrow{\alpha} \Omega^k_{cv}(E|U) \oplus \Omega^k_{cv}(E|V) \xrightarrow{\beta} \Omega^k_{cv}(E|U \cap V) \to 0$$

is exact where $\alpha(\omega) = (\omega|U) \oplus (\omega|V)$ and $\beta(\omega_1 \oplus \omega_2) = (\omega_1|U \cap V) - (\omega_2|U \cap V)$. This induces a long exact sequence

$$\cdots H^k_{cv}(E|U \cup V) \to H^k_{cv}(E|U) \oplus H^k_{cv}(E|V) \to H^k_{cv}(E|U \cap V) \to H^{k+1}_{cv}(E|U \cup V) \cdots$$

The long exact sequence is natural with respect to smooth maps from $U \cup V$ to $U' \cup V'$ which send U to U' and V to V'.

Remark 8.11. An analog of Theorem 8.10 holds where $H^*_{cv}(E)$ is replaced by $H^*_c(E)$, $H^*_{DR}(M)$ is replaced by $H^*_c(M)$, the arrows go the other way and are defined as in Theorem 6.5, and the smooth maps in the concluding sentence are proper. This is a special case of Theorem 6.5.

Theorem 8.12 (Thom Isomorphism). *Let $\pi : E \to M$ be a fiber oriented vector bundle of rank r. Then integration over the fiber induces an isomorphism*

$$\pi_* : H^{k+r}_{cv}(E) \to H^k(M).$$

The inverse of this isomorphism is called the **Thom isomorphism***.*

Proof. There is a good cover such that the bundle is trivial over each finite intersection of elements of the cover. Hence it is enough to prove the theorem for a product bundle $M \times \mathbb{R}^r \to M$ where M is smoothly contractible. The argument in this case does not depend on the fact that M is contractible. In [1] on page 38 the proof is given in case $r = 1$ □

Theorem 8.13 (Thom Isomorphism). *Let $\pi : E \to M$ be a fiber oriented vector bundle of rank r. Then integration over the fiber induces an isomorphism*

$$\pi_* : H^{k+r}_c(E) \to H^k_c(M).$$

The inverse of this isomorphism is called the **Thom isomorphism for compactly supported cohomology***.*

Exercise 8.14. When E and M are both oriented there are isomorphisms

$$H_c^{k+r}(E) = H^{m-k-r}(E)^* = H^{m-k-r}(M)^* = H_c^k(M) \qquad (*)$$

The outer two are Poincaré duality and the one in the middle is because M (or more precisely the zero section) is a deformation retract of E. When E and M are both oriented the bundle $\pi : E \to M$ is fiber oriented so we have two isomorphisms $H_c^{k+r}(E) = H_c^k(M)$, one via $(*)$ and the other via Theorem 8.13. Show they are the same.

Definition 8.15. Let $\pi : E \to M$ be a fiber oriented vector bundle of rank r. A **Thom form** for E is a closed form $\Phi \in \Omega_{cv}^r(E)$ such that $\pi_*\Phi = 1$. The **Thom class** of E is the image of $1 \in H^0(M)$ under the Thom isomorphism $H^*(M) \to H_{cv}^{*+r}(E)$. Thus a closed r-form of compact vertical support is a Thom form iff its cohomology class is the Thom class. Note that the condition that $\Phi \in \Omega_{cv}^r(E)$ is closed implies that $\pi_*\Phi$ is closed and hence locally constant so that (if M is connected) a Thom form is a closed form $\Phi \in \Omega_{cv}^r(E)$ such that $\int_{E_p} \omega = 1$ for some and hence every $p \in M$.

Proposition 8.16. *A closed form $\Phi \in \Omega_{cv}^r(E)$ is a Thom form if and only if the Thom isomorphism is induced by the chain map*

$$\Omega^*(M) \to \Omega_{cv}^{*+r}(E) : \omega \mapsto \pi^*\omega \wedge \Phi.$$

Proof. By the Projection Formula 8.6 we have

$$\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_*\Phi = \omega$$

if $\pi_*\Phi = 1$ so the chain map $\omega \mapsto \pi^*\omega \wedge \Phi$ is a right inverse to the chain map π_* and hence induces the inverse to the isomorphism $H_{cv}^*(E) \to H^{*-r}(M)$ induced by π_*. This inverse is (by definition) the Thom isomorphism. To prove the converse take $\omega = 1$. □

Theorem 8.17. *Let $\Phi \in \Omega_{cv}^r(E)$ be a Thom form for the fiber oriented vector bundle $\pi : E \to M$. Assume that M is oriented and let $m = \dim(M)$ and $i : M \to E$ be the zero section. Then*

$$\int_M i^*\mu = \int_E \mu \wedge \Phi$$

for $\mu \in \Omega_c^m(E)$. In other words, the Thom class is the Poincaré dual of the submanifold $i(M)$ of E.

Proof. The composition $i \circ \pi$ is homotopic to the identity of E so $\pi^*(i^*\mu)$ and μ represent the same element of $H^m(E)$. Hence $\mu = \pi^*i^*\mu + d\tau$ for some $\tau \in \Omega^{m-1}(E)$. Thus

$$\int_E \mu \wedge \Phi = \int_E (\pi^*i^*\mu + d\tau) \wedge \Phi.$$

But

$$\int_E d\tau \wedge \Phi = \int_E d(\omega \wedge \Phi) = 0$$

by Stokes and

$$\int_E \pi^* i^* \mu \wedge \Phi = \int_M i^* \mu \wedge \pi_* \Phi = \int_M i^* \mu$$

by the Projection Formula 8.6. □

Theorem 8.18. *Let $f : N \to M$ be a smooth map and $\pi : E \to M$ be a fiber oriented vector bundle. Recall the commutative diagram*

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

where $F(q, v) = (f(q), v)$. Then

(1) *This diagram respects integration over the fiber, i.e. $F^*\omega \in \Omega^k_{cv}(f^*E)$ for $\omega \in \Omega^k_{cv}(E)$, and $(f^*\pi)_*(F^*\omega) = f^*(\pi_*\omega)$.*

(2) *If $\Phi \in \Omega^e_{cv}(E)$ is a Thom class for E, then $F^*\Phi \in \Omega^e_{cv}(f^*E)$ is a Thom class for f^*E.*

9 Tubular neighborhoods

Definition 9.1. A **fiber Riemannian metric** on a smooth vector bundle $\pi : E \to M$ is a function which assigns to each point $p \in S$ an inner product g_p on the fiber E_p and which is smooth in the sense that the function

$$E \to \mathbb{R} : (p, v) \mapsto g_p(v, v)$$

is smooth. A **Riemannian metric** on a manifold M is a fiber Riemannian metric on the tangent bundle $TM \to M$.

Exercise 9.2. Show that for a trivial bundle $E = M \times \mathbb{R}^n$ a fiber Riemannian metric has form

$$g_p(v, w) = \sum_{i=1}^n \sum_{j=1}^n g_{ij}(p) v_i w_j$$

for $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ where the functions g_{ij} are smooth and the matrix $(g_{ij}(p))$ is positive definite for each p.

Exercise 9.3. Show that a smooth vector bundle admits a fiber Riemannian metric. (Hint: Partitions of unity.)

Exercise 9.4. Let $\pi : E \to M$ be a smooth vector bundle and B be a neighborhood of the zero section. Show that there are neighborhoods B_1 and B_2 of the zero section with $B_1 \subset B_2 \subset B$ and a fiber preserving diffeomorphism $\psi : E \to B_2$ such that $\psi|B_1$ is the identity. (Fiber preserving means $\psi(E_p) = B_2 \cap E_p$ for $p \in M$.)

Exercise 9.5. Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be smooth vector bundles and $\psi : E_1 \to E_2$ be a diffeomorphism such that $\psi(0_{1p}) = 0_{2p}$ for $p \in M$. Show that there is an isotopy $\{\psi_t : E_1 \to E_2\}_{t\in[0,1]}$ such that $\psi_1 = \psi$, $\psi_t(0_{1p}) = 0_{2p}$, and $\psi_0 : E_1 \to E_2$ is a vector bundle isomorphism, i.e. $\psi_0(E_{1p}) = E_{2p}$ and $\psi_0|E_{1p}$ is a linear isomorphism between E_{1p} and E_{2p}. Hint: When M is a point, we may take $\psi_t(v) = t^{-1}\psi(tv)$.

Definition 9.6. A **vector subbundle** of a vector bundle $\pi : E \to M$ is a subset $H \subset M$ such that at every point of M there is a local trivialization $\Psi : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that $\Psi(U \times \mathbb{R}^k \times 0) = H \cap \pi^{-1}(U)$.

Exercise 9.7. Let $\pi : E \to M$ be a vector bundle equipped with a fiber Riemannian metric g and $H \subset E$ be a subbundle. Show that

$$H^\perp := \bigsqcup_{p\in M} H_p^\perp$$

is also a subbundle. Here $H_p^\perp$ is the orthogonal complement to H_p in E_p. Hint: Gram Schmidt.

Definition 9.8. Let S be a smooth submanifold of a manifold M. A **tubular neighborhood** of S in M consists of a vector bundle $\pi : E \to S$, a neighborhood $B \subset E$ of the zero section and a diffeomorphism $\psi : B \to U$ onto a neighborhood U of S in M such that $\psi(0_p) = p$ for $p \in S$.

Exercise 9.9. Let M be a submanifold of $\mathbb{R}^n$. Show that $TM \subset M \times \mathbb{R}^n$ is a subbundle of the trivial vector bundle $M \times \mathbb{R}^n \to M$. Conclude that $T^\perp M$ is a subbundle of the trivial bundle and that $M \times \mathbb{R}^n = TM \oplus T^\perp M$. Here the **tangent bundle** is the set

$$TM := \{(c(0), \dot{c}(0)) \in M \times \mathbb{R}^n : c \in C^\infty(\mathbb{R}, M)\}.$$

Exercise 9.10. Let M be a submanifold of $\mathbb{R}^n$. Show that there is a neighborhood B of $M \times 0$ in $T^\perp M$ such that the map $B \to \mathbb{R}^n : (p, v) \mapsto p + v$ is a tubular neighborhood of M in $\mathbb{R}^n$. Hint: Inverse function theorem.

Exercise 9.11. Assume that M be a closed submanifold of $\mathbb{R}^n$ and let $B \to M$ be the tubular neighborhood of Exercise 9.10. Show that if the neighborhood B is sufficiently small,then for $(p, v) \in B$ the point of M closest to $p + v$ is p. Hint: Minimize $r_q(p) = \|q - p\|^2$.

Exercise 9.12. (Existence of Tubular Neighborhoods.) Let S be a submanifold of M. Show that there is a tubular neighborhood of S in M. Hint: Assume that M is a submanifold of $\mathbb{R}^n$ and use the inverse of the tubular neighborhood constructed in Exercise 9.10.

Exercise 9.13. (Uniqueness of Tubular Neighborhoods.) Let S be a closed submanifold of M and for $i = 1, 2$ let $\pi_i : E_i \to S$ be a vector bundle and $\psi_i : B_i \to U_i \subset M$ be a tubular neighborhood where $B_i \subset E_i$. Show that there is a neighborhood U of S in $U_1 \cap U_2$, a smooth isotopy $\{f_t : M \to M\}_{y \in [0,1]}$, and a vector bundle isomorphism $L : E_1 \to E_2$ such that $f_t(z) = z$ for $z \notin U_1 \cap U_2$, $f_t(p) = p$ for $p \in S$, $f_0 = \mathrm{id}_M$, and $f_1|U = \psi_2 \circ L \circ \psi_1^{-1}|U$. Hint: Modify the isotopy of Exercise 9.5 so that it is stationary away from the zero section.

Exercise 9.14. Let S be a submanifold of M. The **normal bundle** to S in M is the vector bundle $N \to S$ defined by

$$N_p = T_pM/T_pS$$

for $p \in S$. Show that if $\pi : E \to S$ is the domain of a tubular neighborhood of a submanifold S of a manifold M, then E is isomorphic to the normal bundle to S in M.

10 Transversality

Definition 10.1. Let $f : M \to N$ be a smooth map and $W \subset N$ be smooth submanifold. We say f is **transverse** to W iff

$$T_{f(p)}N = (T_pf)(T_pN) + T_{f(p)}N$$

for all $p \in f^{-1}(W)$. Two submanifolds R and S of M are said to be **transverse** iff

$$T_pM = T_pR + T_pS$$

for all $p \in R \cap S$. Note that R and S are transverse iff the inclusion map $R \to M$ is transverse to S.

Theorem 10.2. *Assume that the smooth map $f : M \to N$ is transverse to the submanifold $W \subset N$. Then $f^{-1}(W)$ is a submanifold of M and*

$$T_pf^{-1}(W) = (T_pf)^{-1}T_{f(p)}W$$

for $p \in f^{-1}(W)$. Hence codimension is preserved, i.e.

$$\dim(M) - \dim(f^{-1}(W)) = \dim(N) - \dim(W).$$

Corollary 10.3. *Assume that the submanifolds R and S intersect transversally in M. Then $R \cap S$ is a submanifold of M and*

$$T_p(R \cap S) = (T_pR) \cap (T_pS)$$

for $p \in R \cap S$. Hence

$$\dim(R \cap S) = \dim(R) + \dim(S) - \dim(M).$$

Remark 10.4. When f is transverse to W, the pull back by $f|f^{-1}(W)$ of the normal bundle to W in N is the normal bundle to $f^{-1}(W)$ in M, i.e. the derivative $T_pf : T_pM \to T_pN$ induces an isomorphism

$$T_pM/T_pf^{-1}(W) = T_{f(p)}N/T_{f(p)}W$$

for $p \in M$. Using this isomorphism the normal bundle to $f^{-1}(W)$ in M inherits a fiber orientation from the normal bundle to W in N. In particular, if M, N, and W are oriented, so is $f^{-1}(W)$.

Exercise 10.5. Assume that the submanifolds R and S intersect transversally in M and that $p \in R \cap S$. Show that there is a neighborhood U of p in M and a diffeomorphism $\phi : U \to \mathbb{R}^{m-s} \times \mathbb{R}^{r+s-m} \times \mathbb{R}^{m-r}$ such that $\phi(p) = 0$ and

$$\phi(U \cap R) = \mathbb{R}^{m-s} \times \mathbb{R}^{r+s-m} \times 0 \quad \text{and} \quad \phi(U \cap S) = 0 \times \mathbb{R}^{r+s-m} \times \mathbb{R}^{m-r}.$$

(Hence $\phi(U \cap R \cap S) = 0 \times \mathbb{R}^{r+s-m} \times 0$.)

Remark 10.6. The transverse intersection $R \cap S$ of two oriented submanifolds R an S of an oriented manifold M inherits an orientation as follows. Let $m = \dim(M)$, $r = \dim(R)$, $s = \dim(S)$, $p \in R \cap S$, and $(v_1, \ldots, v_m)$ be a positively oriented frame of T_pM such that $(v_1, \ldots, v_{m-s})$ is a positively oriented frame of T_pR and $(v_{m-r+1}, \ldots, v_m)$ is a positively oriented frame of T_pS. Then $(v_{m-r+1}, \ldots, v_{m-s})$ is a positively oriented frame of $T_p(R \cap S)$.

Definition 10.7. Since the exterior derivative is a wedge derivation the wedge product

$$\Omega^i(M) \times \Omega^j(M) \to \Omega^{i+j}(M) : (\omega, \eta) \mapsto \omega \wedge \eta$$

induces a product

$$H^i_{DR}(M) \times H^j_{DR}(M) \to H^{i+j}_{DR}(M)$$

which we also call the **wedge product**. It corresponds to the **cup product** in singular cohomology.

Definition 10.8. Let $\pi : E \to M$ be a fiber oriented vector bundle. The **Euler class** of E is the pull back

$$e(E) := i^*\Phi$$

of the Thom class of E by the zero section $i : M \to E$.

Theorem 10.9. *Let $E_1 \to M$ and $E_2 \to M$ be fiber oriented vector bundles, $E = E_1 \oplus E_2 \to M$ be the direct sum, $\mathrm{pr}_i : E \to E_i$ be the projections, and $\Phi_i \in \Omega^{r_i}_{cv}(E_i)$ be Thom forms. Then*

$$\Phi := \mathrm{pr}_1^*\Phi_1 \wedge \mathrm{pr}_2^*\Phi_2$$

is a Thom form on E.

Corollary 10.10. *For fiber oriented vector bundles* $E_1 \to M$ *and* $E_2 \to M$ *we have*

$$e(E_1 \oplus E_2) = e(E_1) \wedge e(E_2).$$

Theorem 10.11. *Let* S *be an oriented submanifold of an oriented manifold* M*,* $\pi : E \to S$ *be the domain of a tubular neighborhood* $\psi : E \to M$ *of* S *in* M*,* $\Phi \in \Omega_{cv}^{m-s}(E)$ *be a Thom form and* $\eta \in \Omega^*(M)$ *be determined by the conditions* $\psi^*\eta = \Phi$ *and* $\operatorname{supp}(\eta) \subset \psi(E)$*. Then* η *represents the Poincaré dual of* S*, i.e.*

$$\int_S i^*\omega = \int_M \omega \wedge \eta$$

for $\omega \in \Omega_c^s(M)$ *where* $i : S \to M$ *is the inclusion. (The fiber orientation on* E *is determined by the condition that* ψ *be orientation preserving.)*

Corollary 10.12. *Let* R *and* S *be closed oriented submanifolds of an oriented manifold* M*. Assume that they intersect transversely so that* $R \cap S$ *is a submanifold. Let* $\eta_R \in H_{DR}^{m-r}(M)$*,* $\eta_S \in H_{DR}^{m-s}(M)$*, and* $\eta_{R\cap S} \in H_{DR}^{2m-s-r}(M)$ *be the Poincaré duals of* R*,* S*, and* $R \cap S$ *respectively. Then*

$$\eta_{R\cap S} = \eta_R \wedge \eta_S.$$

Corollary 10.13. *Let* $f : M \to N$ *be a smooth map between oriented manifolds and* $W \subset N$ *be a closed oriented submanifold. Assume* f *is transverse to* W *and equip the submanifold* $f^{-1}(W)$ *of* M *with the orientation of Remark 10.4. Then*

$$\eta_{f^{-1}(W)} = f^*\eta_W$$

where $\eta_W \in H^k(N)$ *and* $\eta_{f^{-1}(W)} \in H^k(M)$ *are the Poincaré duals of* W *and* $f^{-1}(W)$ *respectively.*

Remark 10.14. Assume the hypotheses of Corollary 10.12 and that $\dim R + \dim S = \dim M$ so that the intersection $R \cap S$ is discrete. Let $o(p, R, S, M) = \pm 1$ be the orientation of $R \cap S$ at $p \in R \cap S$ as in Remark 10.6. Then

$$\sum_{p \in R\cap S} g(p)o(p, R, S, M) = \int_M g\eta_{R\cap S} = \int_M g\, \eta_R \cap \eta_S$$

for every function g of compact support.

Lemma 10.15. *Let* v *be a section of a vector bundle* $\pi : E \to M$ *and* $p \in M$ *be a zero of* v*. The* **intrinsic derivative** $Dv(p) : T_pM \to E_p$ *(defined in the proof) is independent of the choice of local trivialization used to define it.*

Proof. Consider two vector bundle charts

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\Phi_i"] \arrow[d, "\pi"] & X_i \times \mathbb{R}^n \arrow[d, "\mathrm{pr}_1"] \\
U_i \arrow[r, "\phi_i"] & X_i
\end{tikzcd}$$

for $i = 1, 2$; i.e. $\phi_i : U_i \to X_i$ is a diffeomorphism from an open subset U_i of M onto an open subset X_i of $\mathbb{R}^m$ and Φ_i is a diffeomorphism which is a linear isomorphism on each fiber. The overlap map $\Phi_2 \circ \Phi_1^{-1}$ has form

$$X_{12} \times \mathbb{R}^n \to X_{21} \times \mathbb{R}^n : (x, v) \mapsto (\phi(x), \Phi(x)v)$$

where $X_{ij} = \phi_i(U_1 \cap U_2) \subset X_i$, $\phi = \phi_2 \circ \phi_1^{-1}$, and $\Phi : X_{12} \to \mathrm{GL}_n(\mathbb{R})$ is smooth. In each chart the section v is represented by a map $v_i : X_i \to \mathbb{R}^n$ and the sections are related by

$$v_2(\phi(x)) = \Phi(x)v_1(x).$$

By the chain rule $Dv_2(\phi(x))\mathbb{D}\phi(x)\hat{x} = D\Phi(x)\hat{x}v_1(x) + \Phi(x)Dv_1(x)$ so

$$Dv_1(x_0) = \Phi(x_0)Dv_1(x_0)D\phi(x_0)^{-1}$$

if $v(p) = 0$ and $\phi_1(p) = x_0$. This is the rule for how a linear map $T_pM \to E_p$ changes coordinates and defines the intrinsic derivative $Dv(p) : T_pM \to E_p$. □

10.16. Assume that $\pi : E \to M$ is a fiber oriented vector bundle over an oriented manifold M and that the rank of E is the same as the dimension m of M, i.e. $\dim E_p = \dim T_pM = \dim M$ for $p \in M$. A zero p of v is called **nondegenerate** iff $v : M \to E$ is transverse to the zero section at p, i.e. $Dv(p)$ is invertible. By the implicit function theorem nondgenerate zeros are isolated. We define the **index** $i(v, p)$ of a nondegenerate zero of v by

$$i(v, p) = \operatorname{sgn} \det Dv(p).$$

It is easy to see that $i(v, p)$ is the degree of the map

$$\mathbb{S}^{m-1} \to \mathbb{S}^{m-1} : x \mapsto \frac{v(x)}{|v(x)|}$$

in local coordinates so chosen that $p = 0$ and 0 is the only zero of v inside $\mathbb{S}^{m-1}$.

Corollary 10.17. *Under the hypotheses of 10.16 we have*

$$\sum_{v(p)=0} i(v, p) = \int_M e(E).$$

for a section v with only nondegenerate zeros.

Proof. This is an instance of the formula $\eta_{R\cap S} = \eta_R \cap \eta_S$ described in Remark 10.14. Take $R = v(M) \subset E$ and S to be the zero section. Let $i : M \to E$ be the zero section so that $S = i(M)$ and $\Phi \in \Omega^*_{cv}m(E)$ be a Thom form. It is easy to check that $i(v, p) = o(R, S, p)$. The Thom form Φ represents the Poincarè dual of both R and S and the pull back $i^*\Phi$ represents the Euler class of E. Take $g = 1$ in Remark 10.14. □

11 The Lefschetz Fixed Point Theorem

11.1. In this section M is an oriented compact manifold of dimension m,

$$\pi, \rho : M \times M \to M$$

are the projections onto the first and second factor respectively, $\{\omega_i\}_i$ is a vector space basis for $H^*_{DR}(M)$, and $\{\tau_j\}_j$ be the dual basis, i.e.

$$\int_M \omega_i \wedge \tau_j = \delta_{ij}.$$

Lemma 11.2. *Let $f : M \to M$ be a smooth map, and*

$$\Gamma := \{(x, f(x)) : x \in M\} \subset M \times M$$

be the graph of f. Define numbers $f_{ij} \in \mathbb{R}$ by

$$f_{ij} := \int_M f^*\omega_i \wedge \tau_j.$$

Then the form

$$\eta_\Gamma := \sum_i (-1)^{\deg \omega_i} f_{ii}\, \pi^*\omega_i \wedge \rho^*\tau_i$$

represents the Poincaré dual of Γ in $M \times M$.

Proof. (From [1] page 127.) We must show that for any form $\lambda \in \Omega^m(M \times M)$ we have

$$\int_\Gamma \lambda = \int_{M\times M} \lambda \wedge \eta_\Gamma.$$

By the Künneth Formula 7.11 the forms $\rho^*\omega_r \wedge \pi^*\tau_s$ represent a basis for $H^*_{DR}(M \times M)$ so it is enough to prove the formula for these. Let $F : M \to \Gamma$ by $F(x) = (x, f(x))$. Then $\pi \circ F = \mathrm{id}_M$ and $\rho \circ F = f$ so

$$\int_\Gamma \rho^*\omega_r \wedge \pi^*\tau_s = \int_M F^*\rho^*\omega_r \wedge F^*\pi^*\tau_s = \int_M f^*\omega_r \wedge \tau_s = f_{rs}.$$

On the other hand

$$\int_{M\times M} \rho^*\omega_r \wedge \pi^*\tau_s \wedge \pi^*\omega_i \wedge \rho^*\tau_j = \pm \int_{M\times M} \pi^*(\omega_i \wedge \tau_r) \wedge \rho^*(\omega_s \wedge \tau_j) = \pm\delta_{ir}\delta_{sj}.$$

Hence

$$\int_{M\times M} \rho^*\omega_r \wedge \pi^*\tau_s \wedge \eta_\Gamma = \pm f_{rs}.$$

Now check the sign. $\square$

Corollary 11.3. *The form*

$$\eta_\Delta := \sum_i (-1)^{\deg \omega_i} \pi^* \omega_i \wedge \rho^* \tau_i \in \Omega^m(M \times M)$$

represents the Poincaré dual of the diagonal

$$\Delta := \{(x, x) : x \in M\}$$

in $M \times M$.

Proof. In Lemma 11.2 take $f = \mathrm{id}_M$ so $f_{ij} = \delta_{ij}$. □

Corollary 11.4. *For a compact oriented manifold* M *the integral of the Euler class* $e(TM)$ *of the tangent bundle is the Euler characteristic*

$$\int_M e(TM) = \chi(M) := \sum_{k-0}^{m} (-1)^k \dim H^k_{DR}(M)$$

Proof. [1] page 128. □

11.5. Assume $f : M \to M$ is a smooth map of a compact oriented manifold, let $\Gamma \subset M \times M$ denote the graph of f and $F : M \to M \times M : x \mapsto (x, f(x))$ denote the graph map so point $p \in M$ is a fixed point of f (i.e. $f(p) = p$) iff $(p, p) \in \Gamma \cap \Delta$ iff $p \in F^{-1}(\Delta)$. Then the graph map $F : M \to M \times M$ is transverse to the diagonal Δ at a fixed point p if and only the submanifolds Γ and Δ of $M \times M$ intersect transversally at (p, p) if and only if the derivative $df(p) : T_pM \to T_pM$ is invertible. In this case we say that the fixed point is **nondegenerate** and define the **fixed point index** of f at p be

$$i(f, p) := o(p, \Delta, \Gamma, M \times M) = \operatorname{sgn} \det(df(p) - \mathrm{id}_{T_pM})$$

where the intersection orientation of $o(p, \Delta, \Gamma, M \times M)$ of Γ and Δ at (p, p) is as in Remark 10.6.

Theorem 11.6 (Lefschetz Fixed Point Theorem). *Assume the situation of 11.5 and that* f *has only nondegenerate fixed points we have*

$$\sum_{f(p)=p} i(f, p) = \sum_{k=0}^{m} (-1)^k \mathrm{trace}(f^* : H^k(M) \to H^k(M)).$$

Proof. [1] page 129. □

Corollary 11.7 (Hopf Index Theorem). *For a vector field* v *a compact oriented manifold with only nondegenerate zeros we have*

$$\sum_{v(p)=0} i(v, p) = \chi(M).$$

12 The Nerve of a Cover

12.1. Let N be finite abstract simplicial complex, i.e. a finite collection N of nonempty finite sets such that $\emptyset \neq B \subset A \in N \implies B \in N$. The elements of N are called **simplices**. A **p-simplex** is a simplex with exactly $p+1$ elements. We denote the set of all p-simplices of N by N_p. We do not distinguish between a 0-simplex $\{\alpha\}$ and it sole element α. A 0-simplex is also called a **vertex**. The **geometric realization** of N is the set $|N|$ of functions $x : |N_0| \to [0,1]$ such that

$$\sum_{\alpha \in N_0} x(\alpha) = 1, \qquad \{\alpha : x(\alpha) \neq 0\} \in N.$$

For each vertex $\alpha \in N_0$ the function $\xi_\alpha : |N| \to [0,1]$ defined by

$$\xi_\alpha(x) = x(\alpha)$$

is called the **barycentric coordinate** associated to the vertex α. The set

$$U_\alpha = \xi_\alpha^{-1}((0,1])$$

is called the **open star** of the vertex α and for $A \in N_p$ the intersection

$$U_A := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}, \qquad A = \{\alpha_0, \alpha_1, \ldots, \alpha_p\}.$$

is called the open star of the simplex A.

Remark 12.2. The barycentric coordinates form a partition of unity, i.e. $\sum_\alpha \xi_\alpha = 1$. The collection

$$\mathcal{U} = \{U_\alpha\}_{\alpha \in N_0}$$

is an open cover of $|N|$. However, the partition of unity is not quite subordinate to the cover as the support of ξ_α is the closure of U_α, not a subset of U_α. A partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to the cover $\mathcal{U} = \{U_\alpha\}_\alpha$ map be viewed as a map $\rho : |N| \to |N|$. (The analog reading $\{\xi_\alpha\}_\alpha$ for $\{\rho_\alpha\}_\alpha$ is the identity map.) The map ρ maps each subcomplex onto itself. It is homotopic to the identity and is hence a homotopy equivalence.

12.3. Now suppose that $\mathcal{U} := \{U_\alpha\}_{\alpha \in \mathcal{N}_0}$ is an arbitrary finite open cover of a space M. For each finite set $A \subset N_0$ introduce the abbreviation

$$U_A := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}, \qquad A = \{\alpha_0, \alpha_1, \ldots, \alpha_p\}.$$

The **nerve** of the cover is the abstract simplicial complex $\mathcal{N}$ consisting of those finite subsets A such that $U_A \neq \emptyset$. Thus

$$\mathcal{N}_p := \{A \subset N_0 : A \in \mathcal{N}, \text{ and } \#(A) = p+1\}$$

denotes the set of p-simplices of N. As usual, we do not distinguish the vertex $\alpha \in N_0$ and the 0-simplex $\{\alpha\}$. It is easy to see that if the cover $\mathcal{U}$ arises from a simplicial complex N as in 12.1 then the simplicial complexes N and $\mathcal{N}$ are isomorphic.

12.4. A **triangulation** of a space M is a homeomorphism $h : |N| \to M$ from the geometric realization of an abstract simplicial complex N onto M. A **smooth triangulation** of a smooth manifold M is a triangulation such that the restriction to each simplex is smooth. (The geometric realization $|B|$ of an n-simplex B of N admits an obvious homeomorphism $\Delta^n \to |B|$ and *smooth* means that the composition of h with this homeomorphism is a smooth map $\sigma_B : |B| \to M$.)

Lemma 12.5. *Let M be a smooth manifold equipped with a smooth triangulation as in 12.4, $\rho : M \to |N|$ be the map associated to a smooth partition of unity subordinate to the cover by open stars as in Remark 12.2, and $\sigma_B : \Delta^n \to M$ a smooth n-simplex as in 12.4. Then*

$$\int_{\sigma_B} \rho^* \omega = \int_{\sigma_B} \omega$$

for any $\omega \in \Omega^n(M)$.

Proof. Replacing ω by $\sigma_B^* \omega$ and ρ by $\rho \circ \sigma_B$ we see that it is enough to prove that

$$\int_{\Delta^n} \rho^* \omega = \int_{\Delta^n} \omega$$

for any smooth n-form on Δ^n and any smooth partition of unity $\{\rho_0, \ldots, \rho_n\}$ with ρ_i supported in $U_i := \{(y_0, \ldots, y_n) \in \Delta^n : y_i \neq 0\}$. The proof is essentially the same as the proof that $\int_M f^* \omega = \deg(f) \int_N \omega$ for a smooth map $f : M \to N$ between compact oriented manifolds of the same dimension and a form ω of top degree on N. By Sard's Theorem it is enough to integrate over the set of regular points of the map ρ. The formula then follows from the fact that the map ρ is onto and has degree one in the sense that the algebraic number of preimages of a regular value is one. (Proof: Exercise.) ☐

13 The Čech deRham Complex

13.1. Throughout this section M is a smooth manifold and

$$\mathcal{U} := \{U_\alpha\}_{\alpha \in N_0}$$

denotes a finite open cover of a smooth manifold M. Assume that the cover $\mathcal{U}$ is good, i.e. that for each $A \in N$ the set U_A is contractible. Denote the nerve (see 12.3) of the cover by N. Define

$$C^p(\mathcal{U}, \Omega^q) := \bigoplus_{A \in N_p} \Omega^q(U_A)$$

and

$$C^p(\mathcal{U}, \mathbb{R}) := \bigoplus_{A \in N_p} \mathbb{R}(U_A)$$

where $\mathbb{R}(U)$ denotes the constant functions on U.

13.2. Fix a linear order on the set N_0 of vertices. For $\alpha \in A \in N_p$ define $s(\alpha, A) = \pm 1$ by

$$s(\alpha, A) = (-1)^i$$

where $A = \{\alpha_0 < \alpha_1 < \cdots < \alpha_p\}$ and $\alpha = \alpha_i$. Define $\delta : C^{p-1}(\mathcal{U}, \Omega^q) \to C^p(\mathcal{U}, \Omega^q)$ by

$$(\delta\omega)_A := \sum_{\alpha \in A} s(\alpha, A)\omega_{A\setminus\alpha}|U_A$$

for $(\omega_A)_A \in \mathbb{C}^{p-1}(\mathcal{U}, \Omega^q)$, i.e.

$$(\delta\omega)_{\alpha_0,\dots\alpha_p} = \sum_{i=0}^{p} \omega_{\alpha_0,\dots\hat{\alpha_i}\dots\alpha_p}$$

where $\hat{\alpha_i}$ means that the index α_i is omitted and in each summand on the right restriction to U_A is understood. The operator $\delta : C^{p-1}(\mathcal{U}, \Omega^0) \to C^p(\mathcal{U}, \Omega^0)$ restricts to an operator $C^{p-1}(\mathcal{U}, \mathbb{R}) \to C^p(\mathcal{U}, \mathbb{R})$. Define $r : \Omega^q(M) \to C^0(\mathcal{U}, \Omega^q)$ by

$$(r\omega)_\alpha := \omega|U_\alpha.$$

Remark 13.3. We use the convention that $\omega_{\alpha_0,\dots\alpha_p} = 0$ if a subscript is repeated and that

$$\omega_{\alpha_0,\dots\alpha_p} = \operatorname{sgn}(\sigma)\, \omega_{\alpha_{\sigma(0)},\dots\alpha_{\sigma(p)}}$$

where σ is the permutation of $\{0, \dots, p\}$ such that $\sigma(0) < \sigma(1) < \cdots < \sigma(p)$.

Lemma 13.4. *The sequence*

$$\Omega^q(M) \xrightarrow{r} C^0(\mathcal{U}, \Omega^q) \xrightarrow{\delta} C^1(\mathcal{U}, \Omega^q) \xrightarrow{\delta} C^2(\mathcal{U}, \Omega^q) \xrightarrow{\delta} \cdots$$

is exact.

Lemma 13.5. *Under the correspondence defined in 12.1 the sequence*

$$C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} \cdots$$

is a chain complex isomorphic to the simplicial cochain complex of the simplicial complex N. Hence the inclusion into the singular chain complex of M induces an isomorphism between the homology $H^*_\delta(\mathcal{U}, \mathbb{R})$ and the singular cohomology $H^*(M, \mathbb{R})$ of M.

13.6. Let $i : C^p(\mathcal{U}, \mathbb{R}) \to C^p(\mathcal{U}, \Omega^0)$ be the direct sum of the inclusions (i.e. identify $\mathbb{R}$ with the constant functions) and define $\tilde{d} : C^p(\mathcal{U}, \Omega^q) \to C^p(\mathcal{U}, \Omega^{q+1})$ by

$$(\tilde{d}\omega)_A = (-1)^p d\omega_A$$

for $(\omega_A)_A \in C^p(\mathcal{U}, \Omega^q)$. As $\tilde{d} = d$ when $p = 0$, the map $r : \Omega^q(M) \to C^0(\mathcal{U}, \Omega^q)$ is a cochain map.

Lemma 13.7. *The sequence*

$$C^p(\mathcal{U},\mathbb{R}) \xrightarrow{i} C^p(\mathcal{U},\Omega^0) \xrightarrow{\tilde{d}} C^p(\mathcal{U},\Omega^1) \xrightarrow{\tilde{d}} C^p(\mathcal{U},\Omega^2) \xrightarrow{\tilde{d}} \cdots$$

is exact.

13.8. The various objects defined above fit together in a **double complex**

$$\begin{tikzcd}
{} & {} & & {} & \\
\Omega^q(M) \arrow[u, "d"'] \arrow[r, "r"] & C^0(\mathcal{U},\Omega^q) \arrow[u, "\tilde{d}"'] \arrow[r, "\delta"] & \cdots \arrow[r] & C^p(\mathcal{U},\Omega^q) \arrow[u, "\tilde{d}"'] \arrow[r, "\delta"] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
\Omega^0(M) \arrow[u, "d"'] \arrow[r, "r"] & C^0(\mathcal{U},\Omega^0) \arrow[u, "\tilde{d}"'] \arrow[r, "\delta"] & \cdots \arrow[r] & C^p(\mathcal{U},\Omega^0) \arrow[u, "\tilde{d}"'] \arrow[r, "\delta"] & \cdots \\
 & C^0(\mathcal{U},\mathbb{R}) \arrow[u, "i"'] \arrow[r, "\delta"] & \cdots \arrow[r] & C^p(\mathcal{U},R) \arrow[u, "i"'] \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

Theorem 13.9. *Define* $D : C^n(\mathcal{U},\Omega) \to C^{n+1}(\mathcal{U},\Omega)$ *by*

$$C^n(\mathcal{U},\Omega) := \bigoplus_{p+q=n} C^p(\mathcal{U},\Omega^q), \qquad D = \delta + \tilde{d}.$$

Then $D^2 = 0$ *and the maps* $r : \Omega^*(M) \to C^*(\mathcal{U},\Omega)$ *and* $i : C^*(\mathcal{U},\mathbb{R}) \to C^*(\mathcal{U},\Omega)$ *are chain equivalences.*

Proof. By Lemma 13.5 the rows of this complex are exact (except for the bottom row). By Lemma 13.7 the columns of this complex are exact (except for the leftmost column). This is enough to imply that the cohomology of the leftmost column is isomorphic to the cohomology of the bottom row. See [1] page 96. □

13.10. As noted in [1] page 96 the isomorphism $H^*_\delta(\mathcal{U},\mathbb{R}) \to H^*_{DR}(M)$ of Theorem 13.9 is induced by the chain map

$$C^*(\mathcal{U},\mathbb{R}) \to \Omega^*(M) : (c_A)_A \mapsto \sum_A c_A \omega_A$$

where

$$\omega_A = \rho_{\alpha_0}\, d\rho_{\alpha_1} \wedge \cdots \wedge d\rho_{\alpha_n}$$

for $A = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ and $\{\rho_\alpha\}_\alpha$ is a partition of unity subordinate to the cover $\mathcal{U} = \{U_\alpha\}_\alpha$.

Theorem 13.11. *Assume the situation of 13.10 and that the open cover arises from a smooth triangulation as explained in 12.4. Then*

$$\int_{\sigma_B} \omega_A = \frac{1}{n!}\delta_{AB}$$

for each pair of simplices A and B of the same dimension. Hence the inverse of the isomorphism $H^*_\delta(\mathcal{U},\mathbb{R}) \to H^*_{DR}(M)$ of Theorem 13.9 is inverse of the composition de Rham isomorphism $I : H^*_{DR}(M) \to H^*(M)$ of Theorem 5.19 with the isomorphism $H^*(M) \to H^*_\delta(\mathcal{U},\mathbb{R})$ between singular cohomology and simplicial cohomology as in 13.5.

Proof. If $A \neq B$, then $A \setminus B \neq \emptyset$. If $\alpha \in A \setminus B$, then $\sigma_B^*\rho_\alpha$ vanishes identically and hence also $\sigma_B^*\omega_A = 0$. (Here $\rho : M \to |N|$ is the map determined by the partition of unity as in Remark 12.2.) If $A = B$, then by Lemma 12.5 we have

$$\int_{\sigma_B} \omega_B = \int_{\sigma_B} \rho^*\omega_B = \int_{\Delta_n} \sigma_B^*\rho^*\omega_0 = \int_{\Delta_n} (\rho \circ \sigma_B)^*\omega_0 = \int_{\Delta^n} \omega_0$$

where $\omega_0 = y_0\,dy_1 \wedge \cdots \wedge dy_n$ and $(y_0, y_1, \ldots, y_n)$ are the restrictions of the coordinate functions on $\mathbb{R}^{n+1}$ to Δ^n. Using the variables $u_k = y_0 + \cdots + y_{k-1}$, i.e. $y_k = u_k - u_{k-1}$, we get

$$\int_{\Delta^n} \omega_0 = \int_{u_n=0}^1 \int_{u_{n-1}=u_n}^1 \cdots \int_{u_1=u_2}^1 u_1\,du_1\,du_2 \cdots du_n$$

which is easy to evaluate. □

14 Products

14.1. Continue the assumptions of 13.1. Recall the de Rham complex $d : \Omega^*(M) \to \Omega^{*+1}(M)$, the simplicial chain complex $\delta : C^*(\mathcal{U},\mathbb{R}) \to C^{*+1}(\mathcal{U},\mathbb{R})$ from 13.2 the double complex $D : C^*(\mathcal{U},\Omega) \to C^{*+1}(\mathcal{U},\Omega)$ from 13.8, the restriction map $r : \Omega^n(M) \to C^0(\mathcal{U},\Omega^n)$ from 13.2, and the inclusion map $i : C^n(\mathcal{U},\mathbb{R}) \to C^n(\mathcal{U},\Omega^0)$ from 13.6. The product

$$C^p(\mathcal{U},\Omega^q) \times C^r(\mathcal{U},\Omega^s) \to C^{p+r}(\mathcal{U},\Omega^{q+s})$$

defined by

$$(\omega \cup \tau)_{\alpha_0,\ldots\alpha_{p+r}} = (-1)^{qr}\omega_{\alpha_0,\ldots\alpha_p} \wedge \tau_{\alpha_p,\ldots\alpha_{p+r}}.$$

satisfies

$$D(\omega \cup \tau) = (D\omega) \cup \tau + (-1)^n \omega \cup (D\tau)$$

where $n = p + q$, $k = r_s$, and therefore induces a product

$$H^n(\mathcal{U},\Omega) \times H^k(\mathcal{U},\Omega) \to D^{n+k}(\mathcal{U},\Omega).$$

Note that the restriction of this product to the subcomplex $C^*(\mathcal{U}, \mathbb{R})$ is the simplicial cup product which induces the usual cup product

$$H^n_\delta(\mathcal{U}, \mathbb{R}) \times H^k_\delta(\mathcal{U}, \mathbb{R}) \to H^{n+k}_\delta(\mathcal{U}, \mathbb{R})$$

on the nerve. The wedge product $\Omega^n(M) \times \Omega^k(M) \to \Omega^{n+k}(M)$ induces a product (also called the **wedge product**)

$$H^n_{DR}(M) \times H^k_{DR}(M) \to H^{n+k}_{DR}(M)$$

in de Rham cohomology.

Theorem 14.2. *The chain equivalences r and i of Theorem 13.9 are ring homomorphisms and hence induce ring isomorphisms $H^*_{DR}(M) \to H^*_D(\mathcal{U}, \Omega)$ and $H^*(\mathcal{U}, \mathbb{R}) \to H^*_D(\mathcal{U}, \Omega)$.*

Proof. See [1] page 174. ☐

Remark 14.3. By Theorem 13.11 the de Rham isomorphism

$$I : H^*_{DR}(M) \to H^*(M, \mathbb{R})$$

of Theorem 5.19 is an isomorphism of rings, i.e. $I(\omega) \cup I(\eta) = I(\omega \wedge \eta)$ for $\omega, \eta \in H^*_{DR}(M)$,

References

[1] R. Bott & L. W. Tu: *Differential Forms in Algebraic Topology*, Springer GTM **82**, 1982.

[2] V. Giuillemin & A. Pollock: *Differential Topology*, Prentic Hall, 1974.

[3] W. S. Massey: *A Basic Course in Algebraic Topology*, Springer GTM **127** 1991.

[4] M. Spivak: *Calculus on Manifolds*, Benjamin (reissued by Perseus) 1965.