The Fundamental Group

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1 The Fundamental Group

1.1. When $\gamma : I \to X$ is a path, we denote by $[\gamma]$ the homotopy class of $\gamma$ (rel $\partial I$); here $\partial I = \{0, 1\}$. Two paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$ determine a path $\alpha \cdot \beta : I \to X$ via the formula

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}; \\ \beta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The fundamental group of the pointed space $(X, x_0)$ is

$$\pi_1(X, x_0) = \{[\gamma] : \gamma(0) = \gamma(1) = x_0\}.$$ 

The group operation is well defined by $[\alpha][\beta] := [\alpha \cdot \beta]$. The identity element is the constant path and the inverse operation is defined by

$$[\gamma]^{-1} := [\gamma], \quad \tilde{\gamma}(t) := \gamma(1 - t).$$

Note that any path $\alpha : I \to X$ gives an isomorphism

$$\pi_1(X, x_0) \to \pi_1(X, x_1) : [\gamma] \mapsto [\tilde{\alpha} \cdot \gamma \cdot \alpha]$$

where $x_0 = \alpha(0), x_1 = \alpha(1)$, and (as before) $\tilde{\gamma}(t) := \gamma(1 - t)$. In this sense, if $X$ is path connected, the fundamental group is independent of the choice of the base point.

Remark 1.2. A groupoid is an algebraic structure consisting of two sets $B$ (the objects) and $G$ (the morphisms) and five maps $s, t : G \to B$ (source and target), $e : B \to G$ (identity), $m : G \times G \to G$ (composition), and $i : G \to G$ (inverse) satisfying the following axioms:

(Identity Law) $g \cdot e_x = g, e_y \cdot g = g$ where $s(g) = x, t(g) = y$.

(Associative Law) $g_1 \cdot (g_1 \cdot g_3) = (g_1 \cdot g_1) \cdot g_3$ for $g_1, g_2, g_3 \in G \times G$.

(Inverse Law) $g \cdot g^{-1} = e_y, g^{-1} \cdot g = e_x$ where $s(g) = x, t(g) = y$. 

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Here we are using the abbreviations

\[ G_s \times_t G := \{(g_1, g_2) \in G \times G : t(g_2) = s(g_1)\}, \]

\[ e_x := e(x), \quad g_1 \cdot g_2 := m(g_1, g_2), \quad g^{-1} := i(g). \]

If the map \( i \) and the Inverse Law are dropped we recover the definition of a category. In a category a morphism which has a (necessarily unique) two-sided inverse is called an isomorphism (or an automorphism if source and target are the same). Thus a groupoid is a category in which every morphism is an isomorphism. For a category \((B, G)\) and \(x, y \in B\) denote the set of morphisms from \(x\) to \(y\) by

\[ G_{x,y} := \{g \in G : s(g) = x, \ t(g) = y\}. \]

The set \( G_x := \{g \in G_{x,x} : g \text{ is an isomorphism}\} \) is called the automorphism group of the object \(x\). With these definitions \(\pi_1(X, x_0)\) is the automorphism group \(x_0\) in the fundamental groupoid defined by

\[ B = X, \quad G = \{[\gamma] \mid \gamma : I \to X\}, \]

\[ s([\gamma]) = \gamma(0), \quad t([\gamma]) = \gamma(1), \quad e(x) = [x] \text{ (the homotopy class of the constant path at the point } x), \quad m([\beta], [\alpha]) = [\alpha \cdot \beta], \text{ and } i(\gamma) = \bar{\gamma}. \]

**Definition 1.3.** A path connected space is called simply connected iff its fundamental group \(\pi_1(X, x_0)\) is trivial.

**Theorem 1.4.** \(\pi_1(S^1) = \mathbb{Z}\) and \(S^n\) is simply connected for \(n > 1\).

## 2 Covering Spaces

**2.1.** A map \(p : E \to B\) is called locally trivial iff every \(x_0 \in B\) there is a neighborhood \(U\) of \(x_0 \in B\) and a homeomorphism \(\phi : U \times F \to p^{-1}(U)\) such that \(p(\phi(x, v)) = x\) for \((x, v) \in U \times F\). The map \(p\) is called the projection, \(E\) is called the total space, \(B\) is called the base, and the topological space \(F\) is called the fiber. It is easy to see that if \(B\) is connected, then the fiber is unique up to homeomorphism. A locally trivial map is also called a fiber bundle, but sometimes the latter term has a more restricted meaning. A covering space is a locally trivial map with discrete fiber. Usually we denote covering spaces with notation like \(p : Y \to X\) or \(p : \tilde{X} \to X\). Note that the projection \(p\) of a covering space is a local homeomorphism.

**Example 2.2.** The archetypal example of a covering space is the exponential map \(p : \mathbb{R} \to S^1\) defined by \(p(\theta) = e^{i\theta}\). The restriction of \(p\) to an open interval is a local homeomorphism but not a covering.

**Example 2.3.** The map \(p : S^1 \to S^1, p(z) = z^n\) is an \(n\)-sheeted covering space.

**Example 2.4.** The various covers of a wedge \(S^1 \vee S^1\) of circles shown on page 58 of [3] are a good source of examples.
2.5. When \( p : \tilde{X} \to X \) and \( f : Y \to X \) we call a map \( \tilde{f} : Y \to \tilde{X} \) a lift of \( f \) iff \( p \circ \tilde{f} = f \). If \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) and \( f : (Y, y_0) \to (X, x_0) \) a pointed lift of \( f \) is a lift \( \tilde{f} \) such that \( \tilde{f}(y_0) = \tilde{x}_0 \).

**Proposition 2.6 (Path Lifting).** Let \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) is a covering space, \( \gamma : \mathbb{I} \to X \), and \( y_0 \in p^{-1}(\gamma(0)) \). Then there is a unique lift \( \tilde{\gamma} : \mathbb{I} \to \tilde{X} \) of \( \gamma \) with \( \tilde{\gamma}(0) = y_0 \).

**Corollary 2.7 (Homotopy Lifting).** Let \( p : \tilde{X} \to X \) is a covering space, \( \{f_t : Y \to X\}_{t \in \mathbb{I}} \) be a homotopy, and \( g : Y \to X \) of be a lift of \( f_0 \). Then there is a unique homotopy \( \{\tilde{f}_t : \tilde{Y} \to \tilde{X}\}_{t \in \mathbb{I}} \) such that \( \tilde{f}_0 = g \) and \( \tilde{f}_t \) is a lift of \( f_t \).

**Proof.** The uniqueness of \( \tilde{f}_t(y) \) follows from 2.6 applied to the path \( \mathbb{I} \to X : t \mapsto f_t(y) \) with starting point \( g(y) \in \tilde{X} \) and this also defines \( \tilde{f}_t(y) \) for \( y \in Y \) and \( t \in \mathbb{I} \). We show that \( (t, y) \mapsto f_t(y) \) is continuous at \( (t_0, y_0) \). By compactness write the interval \( [0, t_0] \) as a union of closed intervals \( [0, t_0] = I_1 \cup \cdots \cup I_k \) so that \( f_t(y_0) \in U_i \) for \( t \in I_i \) where \( U_i \subset X \) is such that \( p \) is trivial over \( U_i \). By induction on \( i \) there are open sets \( U_i \subset \tilde{X} \) such that \( p \) maps \( U_i \) homeomorphically onto \( U_i \) and \( \tilde{f}_t(y_0) \in \tilde{U}_i \) for \( t \in I_i \). By continuity there is a neighborhood \( V \) of \( y_0 \in Y \) such that \( \tilde{f}_t(y) \in U_i \) for \( t \in I_i \) and \( y \in V \). By induction on \( i \), the fact that \( p : \tilde{U}_i \to U_i \) is a homeomorphism, and uniqueness of path lifting, \( \tilde{f}_t(y) \in \tilde{U}_i \) for \( t \in I_i \) and \( y \in V \). Now, since \( p : \tilde{U}_i \to U_i \) is a homeomorphism, the continuity of \( [0, t_0] \times V \to \tilde{X} : (t, y) \mapsto \tilde{f}_t(y) \) at points \( t \in [0, t_0] \) and \( y \in V \) follows from the continuity of \( [0, t_0] \times V \to \tilde{X} : (t, y) \mapsto \tilde{f}_t(y) \)

**Corollary 2.8 (Injectivity).** Let \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) is a covering space. Then the induced map \( p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \) is injective.

**Theorem 2.9 (Lifting Criterion).** Assume \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a covering space and let \( f : (Y, y_0) \to (X, x_0) \) be a map. Assume \( Y \) is path connected and locally path connected. Then

(i) Two pointed lifts of the same map \( f \) are equal.

(ii) There is a pointed lift of \( f \) if and only if \( f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0) \).

**Remark 2.10.** Since \( p_* \) is injective, we have \( f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0) \) if and only if there is a homomorphism \( h : \pi_1(Y, y_0) \to \pi_1(\tilde{X}, \tilde{x}_0) \) such that \( p_* \circ h = f_* \). When these equivalent conditions hold we have \( h = \tilde{f}_* \) as follows: If \( p_* \circ h = f_* \), then \( p_* \circ h = f_* = p_* \circ \tilde{f}_* \) so \( h = \tilde{f}_* \) by the injectivity of \( p_* \). Thus, in the two diagrams

\[
\begin{array}{ccc}
(\tilde{X}, \tilde{x}_0) & \xrightarrow{\tilde{f}} & (Y, y_0) \\
\downarrow p & & \downarrow f \\
(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\
\end{array}
\quad
\begin{array}{ccc}
(\tilde{X}, \tilde{x}_0) & \xrightarrow{\tilde{h}} & \pi_1(\tilde{X}, \tilde{x}_0) \\
\downarrow p_* & & \downarrow p_* \\
\pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, x_0) \\
\end{array}
\]

we can find \( \tilde{f} \) so that the diagram on the left commutes if and only if we can find \( h \) so that the diagram on the right commutes.
Proof of Theorem 2.9. When \( \hat{f} \) is a pointed lift of \( f \) and \( \alpha \) is a path from \( y_0 \) to \( y \), then \( f \circ \alpha \) is a path from \( f(y_0) \) to \( f(y) \) and \( f \circ \alpha \) is a path from \( \hat{f}(y_0) \) to \( \hat{f}(y) \), so uniqueness of the lift follows from uniqueness of path lifting. 'Only if' in part (ii) follows from \( (p \circ f)_* = p_* \circ f_* \). We prove 'if'.

Assume that \( f_* \pi_1(Y, y_0) \subset p_* \pi_1(X, \hat{x}_0) \). As just noted, we must define \( \hat{f}(y) \) by \( \hat{f}(y) = \bar{\gamma}(1) \) where \( \bar{\gamma} \) is the lift of \( \gamma := f \circ \alpha \) with \( \bar{\gamma}(0) = \hat{x}_0 \) and \( \alpha \) is a path from \( y_0 \) to \( y \) in \( Y \). We must show that \( \hat{f} \) is well defined (i.e. independent of the choice of \( \alpha \)) and continuous.

Suppose that \( \beta \) is another path from \( y_0 \) to \( y \). Then \( [\alpha \beta] \in \pi_1(Y, y_0) \) so \( [(f \circ \alpha)(f \circ \beta)] \in f_* \pi_1(Y, y_0) \subset p_* \pi_1(X, \hat{x}_0) \) so there is an element of \( \bar{\gamma} \in \pi_1(X, \hat{x}_0) \) which projects to \( [(f \circ \alpha)(f \circ \beta)] \), i.e. \( p \circ \bar{\gamma} \) is homotopic to \( (f \circ \alpha)(f \circ \beta) \) with endpoints fixed so the lift \( \bar{\gamma} \) of which starts at \( \hat{x}_0 \) has the same endpoint as the lift of \( (f \circ \alpha)(f \circ \beta) \) of which starts at \( \hat{x}_0 \). But this endpoint is \( \hat{x}_0 \) as \( \bar{\gamma} \) is closed. Hence the lift of \( (f \circ \alpha)(f \circ \beta) \) of which starts at \( \hat{x}_0 \) is closed so the lifts of \( f \circ \alpha \) and \( f \circ \beta \) which start at \( \hat{x}_0 \) have the same endpoint as required.

Now we show that \( \hat{f} \) is continuous. Choose \( y_1 \in Y \) and a neighborhood \( W \) of \( \hat{f}(y_1) \) in \( X \). We must construct a neighborhood \( V \) of \( y_1 \) such that \( f(V) \subset W \). As \( p \) is a covering we may shrink \( W \) so that \( p \) maps \( W \) homeomorphically onto a neighborhood \( W' \) of \( f(y_1). \) By the continuity of \( f \) there is a neighborhood \( V \) of \( y_1 \) such that \( f(V) \subset U \). By local path connectedness there is a path connected neighborhood \( U \) of \( y_1 \) with \( U \supset V \). For \( y \in V \) there is a path \( \beta \) from \( y_1 \) to \( y \) in \( V \). Then \( \beta \) lies in \( U \) so \( f \circ \beta \) lies in \( W \) so the lift of \( f \circ \beta \) starting at \( \hat{f}(y_1) \) lies in \( W \). As \( Y \) is path connected there is a path \( \alpha \) from \( y_0 \) to \( y_1 \). Then \( \alpha \beta \) is a path from \( y_0 \) to \( y \) and the lift of \( (f \circ \alpha)(f \circ \beta) \) starting at \( f(y_0) \) ends the same point as the lift of \( f \circ \beta \) starting at \( \hat{f}(y_1) \). This lift is \( \hat{f}(y) \) so \( \hat{f}(y) \) lies in \( W \) as required. \( \square \)

2.11. Denote by \( \text{Cov}(X, x_0) \) the category whose objects are covering spaces \( p : (Y, y_0) \to (X, x_0) \). A morphism in this category from the covering space \( p : (Y, y_0) \to (X, x_0) \) to the covering space \( q : (Z, z_0) \to (X, x_0) \) is a map \( \phi : (Y, y_0) \to (Z, z_0) \) such that \( q \circ \phi = p \). For any group \( G \) let \( \text{Sub}(G) \) denote the category whose objects are subgroups \( H \) of \( G \) and where the morphisms are inclusions \( K \subset H \). There is a functor

\[
\text{Cov}(X, x_0) \to \text{Sub}(\pi_1(X, x_0))
\]

which assigns the subgroup \( p_* \pi_1(Y, y_0) \) to the object \( p : (Y, y_0) \to (X, x_0) \) and assigns the inclusion

\[
p_* \pi_1(Y, y_0) \subset p_* \phi_* \pi_1(Z, z_0) = q_* \pi_1(Z, z_0)
\]

to the morphism \( \phi : (Y, y_0) \to (Z, z_0) \)

Corollary 2.12. Let \( p : (Y, y_0) \to (X, x_0) \) and \( q : (Z, z_0) \to (X, x_0) \) be pointed covering spaces over the same pointed space \( (X, x_0) \). Then

(i) There is at most one morphism from \( p \) to \( q \).
(ii) There is a morphism from \( p \) to \( q \) if and only if \( p_*\pi_1(Y, y_0) \subset q_*\pi_1(Z, z_0) \).

Proof. This is a special case of Theorem 2.9.

**Exercise 2.13.** Let \( \phi : Y \to Z \) be a morphism from the covering space \( p : Y \to X \) to the covering space \( q : Z \to X \), as in 2.11. Assume that \( X \) (and hence also \( Y \) and \( Z \)) is locally connected. Show that \( \phi : Y \to Z \) is a covering space. Give an example showing that the hypothesis that \( X \) is locally connected cannot be dropped.

**Solution.** Choose \( z_0 \in Z \) and let \( x_0 = q(z_0) \). We must find a neighborhood \( V \) of \( z_0 \) and a trivialization of the restriction \( \phi \) to \( \phi^{-1}(V) \). Trivialize \( p \) and \( q \) over a neighborhood of \( x_0 \) and then shrink this neighborhood so it is connected. Restricting to this neighborhood we may assume w.l.o.g. that

\[
Y = X \times A, \quad Z = X \times B, \quad p(x,a) = q(a,b) = x
\]

where \( A \) and \( B \) are discrete and \( X \) is connected. The condition \( q \circ \phi = p \) implies that \( \phi(x,a) = (x, f(x,a)) \) where \( f : X \times A \to B \) is continuous. Let \( z_0 = (x_0, b_0) \) and \( C = \{ a \in A : f(x_0, a) = b_0 \} \). Let \( V = X \times \{ b_0 \} \). Then for \( (x,a) \in X \times A \) we have

\[
\phi(x,a) \in V \iff f(x,a) = b_0 \iff f(x_0,a) = b_0 \iff a \in C
\]

where the middle \( \iff \) holds because \( f \) is locally constant and \( X \) is connected. Thus \( \phi^{-1}(V) = X \times C \) so the inclusion \( X \times C \to X \times A \) is a local trivialization.

Consider \( X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \} \), \( A = \mathbb{N} \), \( B = \{1, 2, \} \), and \( f(x,n) = 1 \) or \( 2 \) according as \( nx < 1 \) or \( xn \geq 1 \). Then the fiber \( \phi^{-1}(x,1) \) is finite for \( x \neq 0 \) and infinite for \( x = 0 \) so \( \phi \) is not trivial over any neighborhood of \( (0,1) \).

2.14. A covering space \( p : \tilde{X} \to X \) of a path connected space \( X \) is called **universal** if \( \tilde{X} \) is connected and simply connected. A space \( X \) is called **semi locally simply connected** if every point \( x_0 \in X \) has arbitrarily small neighborhoods \( U \) such that the homomorphism \( \pi_1(U, x_0) \to \pi_1(X, x_0) \) is trivial.

**Theorem 2.15.** A path connected space and locally path connected space \( X \) has a universal cover if and only if it is semi locally simply connected.

Proof. For “only if” assume that \( p : \tilde{X} \to X \) is universal. Choose \( x_0 \in X \) and a neighborhood \( V \) of \( x \). Choose a neighborhood \( U \) of \( x_0 \) in \( V \) so that \( P \) is trivial over \( U \). Let \( \tilde{U} \subset \tilde{X} \) map homeomorphically to \( U \) by \( p \) and \( \tilde{x}_0 \in \tilde{U} \) be defined by \( p(\tilde{x}_0) = x_0 \). The inclusion \( \pi_1(\tilde{U}, \tilde{x}_0) \to \pi_1(X, x_0) \) factors as \( \pi_1(\tilde{U}, \tilde{x}_0) \to \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \) so it is trivial as \( \pi_1(\tilde{X}, \tilde{x}_0) \) is trivial.

For “if” choose \( x_0 \in X \) and define \( p : \tilde{X} \to X \) by

\[
\tilde{X} := \{ [\gamma] : \gamma : I \to X, \gamma(0) = x_0 \}, \quad p([\gamma]) = \gamma(1).
\]
For each \([\gamma] \in \tilde{X}\) and each path connected neighborhood \(U\) of \(\gamma(1)\) such that \(\pi_1(U, \gamma(1)) \to \pi_1(X, \gamma(1))\) is trivial define
\[
U_{[\gamma]} := \{[\gamma \eta] : \eta : I \to U, \eta(0) = \gamma(1)\}.
\]
(The set \(U_{[\gamma]}\) is clearly independent of the choice of the representative \(\gamma\) of \([\gamma]\).) These sets form a basis for a topology on \(\tilde{X}\). If \(U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset\) then \([\gamma] = [\gamma']\) so \(U_{[\gamma]} = U_{[\gamma']}\). This defines a homeomorphism
\[
U \times \{[\gamma] : \gamma(1) = x_0\} \to p^{-1}(U)
\]
so \(p : \tilde{X} \to X\) is a covering. A function \(\lambda : I \to \tilde{X}\) is continuous if and only if there is a (continuous) map \(\Lambda : I \times I \to X\) with \(\lambda(s) = [\Lambda(s, \cdot)]\) and \(\Lambda(s, 0) = x_0\) for \(s \in I\). Let \(\tilde{x}_0\) be the class of the constant path at \(x_0\). Then the path \(\lambda\) begins and ends at \(\tilde{x}_0\) if and only if there is a map \(\Lambda\) with \(\Lambda(0, t) = \Lambda(1, t) = x_0\) for \(t \in I\). The set \(\{0, 1\} \times I \cup I \times \{0\}\) is a deformation retract of the square \(I \times I\). Composing \(\Lambda\) with this deformation gives a homotopy from \(\lambda\) to a constant loop. Hence \(\pi_1(\tilde{X}, \tilde{x}_0)\) is trivial. For details see 3 page 64.

Exercise 2.16. Show that if two locally path connected spaces are homotopy equivalent, then so are their universal covers. (Assume that the spaces are semi locally simply connected so that they have universal covers.)

3 Group Actions

3.1. A left action (resp. right action) of a group \(G\) on a set \(X\) is a homomorphism (resp. anti-homomorphism) \(G \to \text{Perm}(X)\) where \(\text{Perm}(X)\) denotes the group of permutations of \(X\), i.e. the group of all bijections \(g : X \to X\). For a left action it is customary to write the evaluation map as \(G \times X \to X : (g, x) \mapsto gx\) whereas for a right action we write \(G \times X \to X : (g, x) \mapsto xg\). This notation makes the homomorphism property look like the associative law. An action is called effective iff the homomorphism \(G \to \text{Perm}(X)\) is injective; for an effective action we view \(G\) as a subgroup of \(\text{Perm}(X)\). An action is called free iff no element of \(G\) other than the identity has a fixed point, i.e. iff \(gx = x \implies g = \text{id}\). When \(X\) is a topological space and \(G\) is a topological group (meaning that \(G\) is a topological space and the group operations are continuous) it is required that the evaluation map be continuous; it then follows the action takes values in the subgroup homeomorphism subgroup \(\text{Homeo}(X) \subset \text{Perm}(X)\) of homeomorphisms from \(X\) onto itself. In this section the group \(G\) will always have the discrete topology, i.e. every subset is open.

3.2. An action of \(G\) on \(X\) is called transitive iff for every pair of points \(x, y \in X\) there exists \(g \in G\) with \(gx = y\) (or \(xg = y\) for a right action). For each \(x \in X\) the subgroup of all \(g \in G\) which satisfy \(gx = x\) is called the stabilizer group (some authors say isotropy group) of \(x\) and is denoted \(G_x\). When the action is transitive the map
\[
G/G_x \to X : gG_x \mapsto gx
\]
is well defined and bijective. Here \( G/H \) denotes the set of left cosets \( gH \) of the subgroup \( H \subset G \): the group \( G \) acts on the set \( G/H \) on the left. The map \( G/Gx \to X \) intertwines the two actions. As a right action determines a left action and vice versa (precompose with the anti-automorphism \( g \mapsto g^{-1} \)), all this holds for right actions as well. We denote the set of right cosets of \( H \) by \( H\backslash G \).

### 3.3. Let \( p : Y \to X \) be a covering space and assume that \( Y \) and \( X \) are path connected. For \( x_0 \in X \) the fundamental group \( \pi_1(X, x_0) \) acts on the fiber \( p^{-1}(x_0) \) on the right by the rule \( y_0[\gamma] = y_1 \) iff the lift \( \tilde{\gamma} \) of \( \gamma \) with \( \tilde{\gamma}(0) = y_0 \) satisfies \( \tilde{\gamma}(1) = y_1 \). As \( Y \) is path connected this action is transitive. The stabilizer group of the point \( y_0 \in p^{-1}(x_0) \) is precisely \( p_\ast \pi_1(Y, y_0) \) so there is a bijection

\[
p_\ast \pi_1(Y, y_0) \backslash \pi_1(X, x_0) \to \pi_1^{-1}(x_0)
\]

between the set of right cosets of \( p_\ast \pi_1(Y, y_0) \) in \( \pi_1(X, x_0) \) and the fiber \( \pi_1^{-1}(x_0) \).

We may summarize this as: \textit{the cardinality of the fiber is the index of the subgroup}.

### 3.4. An automorphism of a covering space \( p : Y \to X \) is called a deck transformation; we denote the group of all deck transformations of the cover by \( \text{Aut}(p) \). Thus an element of \( \text{Aut}(p) \) is a homeomorphism \( f : Y \to Y \) such that \( p \circ f = f \).

### 3.5. A group \( G \) acts on the set \( \text{Sub}(G) \) of its subgroups via the adjoint action,

\[
\text{Sub}(G) \to \text{Sub}(G) : H \mapsto gHg^{-1}
\]

for \( g \in G \). The stabilizer of a subgroup \( H \) under this action is called the normalizer of \( H \) in \( G \) and denoted

\[
N(H, G) := \{g \in G : gHg^{-1} = H\}.
\]

It is the largest subgroup of \( G \) containing \( H \) as a normal subgroup; the subgroup \( H \) is a normal subgroup of \( G \) if and only if \( N(H, G) = G \).

### Remark 3.6. If \( G \) acts on \( X \), \( H \) is a subgroup of \( G \), and

\[
X_H := \{x \in X : G_x = H\},
\]

denotes the set of points in \( X \) having stabilizer group \( H \), then

\[
N(H, G) = \{g \in G : g(X_H) = X_H\}
\]

and the formula

\[
(G/N) \times X_H \to X_H : (gH, x) \mapsto gx, \quad N := N(H, G)
\]

gives a well defined action of \( G/N \) on \( X_H \).
Theorem 3.7. Let \( p : Y \to X \) be covering space with \( X \) and \( Y \) path connected and locally path connected and assume \( x_0 \in X \) and \( y_0 \in p^{-1}(x_0) \). Abbreviate \( N := N(p, \pi_1(Y, y_0), \pi_1(X, x_0)) \). Then the groups \( \text{Aut}(p) \) and \( N/p, \pi_1(Y, y_0) \) are anti isomorphic.

Corollary 3.8. The following are equivalent:

(i) The group \( \text{Aut}(p) \) acts transitively on the fiber \( p^{-1}(x_0) \).
(ii) The subgroup \( p_* \pi_1(Y, y_0) \) is a normal subgroup of \( \pi_1(X, x_0) \).
(iii) For every element \( [\gamma] \in \pi_1(X, x_0) \) either every lift of \( \gamma \) is closed or no lift is closed.

When these equivalent conditions hold the covering is called normal (regular by some authors).

Remark 3.9. For a normal covering \( p : Y \to X \) and \( y_0 \in p^{-1}(x_0) \), the groups \( \text{Aut}(p) \) and \( \pi_1(X, x_0)/p_*\pi_1(Y, y_0) \) are anti isomorphic. In particular, for the universal cover \( p \) the groups \( \text{Aut}(p) \) and \( \pi_1(X, x_0) \) are anti isomorphic.

3.10. Let a discrete group \( G \) act on the right on a topological space \( X \) and \( X/G \) denote the orbit space, i.e.

\[
X/G := \{xG : x \in X\}, \quad xG := \{xg : g \in G\}.
\]

Then the projection \( X \to X/G \) (with the quotient topology) is a covering space if and only if every point \( x \in X \) has a neighborhood \( U \) such that \( (Ug_1) \cap (Ug_2) = \emptyset \) for \( g_1 \neq g_2 \). When \( X \to X/G \) is a covering, the group \( G \) is (anti isomorphic to) the group of deck transformations so the covering is normal. Conversely, if \( p : X \to Y \) is a normal covering and \( G = \text{Aut}(p) \) then \( Y \cong X/G^{\text{op}} \). (Note. The group \( \text{Aut}(p) \) is a subgroup of the homeomorphism group and acts on the left. To get a right action we replace \( G \) by \( G^{\text{op}} \) where the opposite group \( G^{\text{op}} \) of \( G \) has the same underlying set as \( G \) and the product of \( a \) and \( b \) in \( G^{\text{op}} \) is product of \( b \) and \( a \) in \( G \). Thus the map \( G \to G^{\text{op}} : g \mapsto g^{-1} \) is an isomorphism.)

Theorem 3.11 (Classification Theorem). Let \( (X, x_0) \) be a path connected pointed space which admits a universal cover. Then the functor described in \([2.11]\) is an equivalence of categories as explained in the proof below.

Proof. This means that

(i) For every subgroup \( H \subset \pi_1(X, x_0) \) there is a covering \( p : (Y, y_0) \to (X, x_0) \) with \( H = p_* \pi_1(Y, y_0) \).
(ii) If \( p : (Y, y_0) \to (X, x_0) \) and \( q : (Z, z_0) \to (X, x_0) \) are connected covering spaces of \( (X, x_0) \) satisfying \( p_* \pi_1(Y, y_0) \subset q_* \pi_1(Z, z_0) \), then there exists a unique map \( \phi : (Y, y_0) \to (Z, z_0) \) with \( q \circ \phi = p \).
(iii) If \( p_* \pi_1(Y, y_0) = q_* \pi_1(Z, z_0) \), then \( \phi \) is a homeomorphism and hence an isomorphism of pointed coverings.
Part (ii) is a restatement of Corollary 2.12 and part (iii) follows immediately from the uniqueness. For part (i) we construct the inverse functor. Let \( u: \tilde{X} \to X \) be a universal cover. Abbreviate \( G := \pi_1(X, x_0) \) and choose \( \tilde{x}_0 \in u^{-1}(x_0) \). The group \( G \) acts on \( \tilde{X} \) on the right by path lifting starting at \( \tilde{x}_0 \). The projection \( u \) induces a homeomorphism from the orbit space \( \tilde{X}/G \) to \( X \). A subgroup \( H \subset G \) determines a covering \( p_H: \tilde{X}/H \to \tilde{X}/G \cong X \).

An inclusion \( K \subset H \subset G \) of subgroups determines a commutative diagram

\[
\begin{array}{ccc}
\tilde{X}/K & \overset{f_{K,H}}{\longrightarrow} & \tilde{X}/H \\
\downarrow{p_K} & & \downarrow{p_H} \\
\tilde{X}/G = X & & \\
\end{array}
\]

\[
\Box
\]

**Example 3.12.** The universal cover of a wedge \( X = S^1 \lor S^1 \) of circles is an infinite tree with four edges at each vertex. Various other coverings of \( X \) are shown in [3] on page 58.

**Example 3.13.** A graph is a one dimensional cell complex. A contractible graph is called a tree. A connected graph \( X \) contains a maximal tree \( A \) and for any maximal tree \( A \) in \( X \) we have that \( X/A \) is a wedge of circles. Now \( (X, A) \) is a cellular pair and \( A \) is contractible so the projection \( X \to X/A \) is a homotopy equivalence. Thus the fundamental group \( \pi_1(X) \) is free for any graph \( X \). This implies that a subgroup of a free group is free as follows. Let \( F \) be a free group and \( X \) be a wedge of circles with one circle for every generator of \( X \) so that \( F = \pi_1(X) \). For any subgroup \( G \) of \( F \) there is a connected cover \( p: Y \to X \) with \( p_*\pi_1(Y) = G \). As \( p_* \) is injective it follows that \( G \) is free.

**3.14.** Now assume that \( X \) is connected and locally path connected and that \( p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) is a universal cover. Then \( X = \tilde{X}/G \) where \( G = \pi_1(X, x_0) = \text{Aut}(p) \). Given any discrete space \( F \) and any left action \( \rho: G \to \text{Perm}(F) \) define

\[
\tilde{X} \times_\rho F := \{[\tilde{x}, v]: \tilde{x} \in \tilde{X}, v \in F \}, \quad [\tilde{x}, v] := \{(\tilde{x}g, \rho(g)^{-1}v) : g \in G \}.
\]

It is not hard to prove that \( \tilde{X} \times_\rho F \to X : [\tilde{x}, v] \mapsto p(\tilde{x}) \) is a covering space, connected if and only if the action on \( F \) is transitive, and that this operation determines a bijective correspondence between isomorphism classes of left actions and isomorphism classes of coverings. (See [3] page 68.)

**Remark 3.15.** This construction is a special case of a more general construction in the theory of fiber bundles as follows. A principal fiber bundle is a fiber bundle \( \pi: P \to B \) such that the total space \( P \) is equipped with a free transitive right action of a topological group \( G \) whose orbits are the fibers of \( \pi \). Given any left action \( \rho: G \to \text{Homeo}(F) \) on a topological space \( F \) the associated fiber bundle \( P \times_\rho F \to B : [u, \xi] \mapsto \pi(u) \) is defined by

\[
P \times_\rho F := \{[u, \xi]: u \in P, \xi \in F \}, \quad [u, \xi] := \{ug, \rho(g)^{-1}\xi) : g \in G \}.
\]
Exercise 3.16. The **projective plane** is the quotient $P^2 = S^2/G$ where $G = \{\pm \text{id}_{S^2}\}$. Find all coverings of $P^2 \times P^2$.

Exercise 3.17. The **commutator subgroup** of a group $G$ is the subgroup $[G, G]$ of $G$ generated by all commutators $[a, b] := aba^{-1}b^{-1}$ with $a, b \in G$. It is a normal subgroup; the quotient group $G/[G, G]$ is called the **Abelianization** of $G$. Show that the Abelianization is characterized by the following universal mapping property: For every homomorphism $\phi : G \to A$ where $A$ is an Abelian group, there is a unique homomorphism $\psi : G/[G, G] \to A$ such that $\psi \circ \pi = \phi$ where $\pi : G \to G/[G, G]$ is the projection.

Exercise 3.18. Define $g_{m,n} : \mathbb{R}^2 \to \mathbb{R}^2$ by $g_{m,n}(x, y) = (x + m, (-1)^m y + n)$ and let $G = \{g_{m,n} : m, n \in \mathbb{Z}\}$. The quotient $\mathbb{R}^2/G$ is called the **Klein bottle**. Determine all double coverings (i.e. the cardinality of the fiber is two) of the Klein bottle. Hint: Compute the Abelianization of $G$.

### 4 Van Kampen’s Theorem

**Definition 4.1.** Let $\{G_{\alpha}\}_{\alpha \in \Lambda}$ be an indexed collection of groups. The **free product** of this collection is the set of all equivalence classes of words (finite sequences) in the disjoint union $\bigsqcup_{\alpha \in \Lambda} G_{\alpha}$ where the equivalence relation is generated by the following operations:

1. adding or deleting an identity element so that
   
   $$w_1 \cdot \cdots w_k e_{\alpha} w_{k+1} \cdots w_n \sim w_1 \cdot \cdots w_k w_{k+1} \cdots w_n$$

   where $e_{\alpha}$ is the identity element of $G_{\alpha}$; and

2. multiplying adjacent elements if they belong in the same group so that

   $$w_1 \cdot \cdots w_k w_{k+1} \cdots w_n \sim w_1 \cdot \cdots (w_k w_{k+1}) \cdots w_n$$

   if $w_k, w_{k+1} \in G_{\alpha}$.

The free product $G$ of the collection $\{G_{\alpha}\}_{\alpha \in \Lambda}$ is often denoted by

$$G = \prod_{\alpha \in \Lambda}^* G_{\alpha}$$

(Hatcher [3] uses the notation $G = \ast_{\alpha} G_{\alpha}$) and when $\Lambda = \{1, 2, \ldots, n\}$ is finite the free product is often denoted by

$$G = G_1 \ast G_2 \ast \cdots \ast G_n.$$ 

The free product $G$ is a group: the group operation is induced by catenation and the identity element is the equivalence class of the empty word. When each $G_{\alpha}$ is isomorphic to $\mathbb{Z}$, the free product is the free group with one generator for each index $\alpha \in \Lambda$. 

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4.2. Below we will consider an indexed collection of groups \( \{G_\alpha\}_{\alpha \in \Lambda} \) and a doubly indexed collection

\[
\{i_{\alpha\beta} : G_{\alpha\beta} \to G_\alpha\}_{\alpha, \beta \in \Lambda}
\]

of group homomorphisms such that \( G_{\alpha\alpha} = G_\alpha, \ i_{\alpha\alpha} = \text{id}_{G_\alpha} \), and \( G_{\alpha\beta} = G_{\beta\alpha} \).

To such a system we associate a subgroup \( N \) of the free product \( G \); it is the normal subgroup generated by all elements of form \( i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1} \). We call the normal subgroup \( N \) of \( G \) the **Van Kampen subgroup** of the system. (When all the \( G_{\alpha\beta} \) for \( \alpha \neq \beta \) are the same the quotient \( G/N \) is commonly called the **amalgamated product**.)

**Theorem 4.3 (Van Kampen).** Assume \( X \) is a space, \( x_0 \in X \), and

\[
X = \bigcup_{\alpha \in \Lambda} A_\alpha
\]

where each \( A_\alpha \) is open and path connected and contains the base point \( x_0 \). Let \( G \) be the free product of the indexed collection \( \{\pi_1(A_\alpha, x_0)\}_{\alpha \in \Lambda} \) of groups and \( N \) be the Van Kampen subgroup determined by the doubly indexed collection

\[
i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)
\]

of homomorphisms induced from the inclusions \( (A_\alpha \cap A_\beta, x_0) \to (A_\alpha, x_0) \). Then

(i) If the double intersections \( A_\alpha \cap A_\beta \ (\alpha, \beta \in \Lambda) \) are path connected, then the natural homomorphism \( G \to \pi_1(X, x_0) \) is surjective.

(ii) If the triple intersections \( A_\alpha \cap A_\beta \cap A_\gamma \ (\alpha, \beta, \gamma \in \Lambda) \) are path connected, then the kernel of this homomorphism is \( N \) so that \( \pi_1(X, x_0) \) is isomorphic to \( G/N \).

**Proof.** Let \( f : ([1], \partial [1]) \to (X, x_0) \). Choose a partition \( 0 = a_0 < a_1 < \cdots < a_n = 1 \) so fine that for each \( k \) there is an index \( \alpha_k \in \Lambda \) with \( f([a_k-1, a_k]) \subset A_{\alpha_k} \). Let \( f_k = f|_{[a_{k-1}, a_k]} \). For \( k = 1, \ldots, n-1 \) choose a path \( g_k \) from \( x_0 \) to \( f(a_k) \) lying in \( A_{\alpha_k} \cap A_{\alpha_{k+1}} \) and take \( g_0 \) and \( g_n \) to be the constant loop at \( x_0 \). Then

\[
[f] = [g_0 f_1 g_1][g_1 f_2 g_2] \cdots [g_{n-2} f_{n-1} g_{n-1}][g_{n-1} f_n g_n]
\]

where \([g_k f_k g_k] \) is the image in \( \pi_1(X, x_0) \) of an element of \( \pi_1(A_{\alpha_k}, x_0) \). This proves (i).

To prove (ii) choose an element \([f_1]_{\alpha_1}[f_2]_{\alpha_2} \cdots [f_m]_{\alpha_m} \in G \) with \([f_j]_{\alpha_j} \in \pi_j(A_{\alpha_j}, x_0) \). Let \( F : \mathbb{I}^2 \to X \) be a homotopy from the loop \( f = f_1 f_2 \cdots f_m \) to the constant loop at \( x_0 \), i.e. \( F(0, t) = f(t) \) and \( F(s, 0) = F(s, 1) = F(1, t) = x_0 \). We must show that \([f_1]_{\alpha_1}[f_2]_{\alpha_2} \cdots [f_m]_{\alpha_m} \in N \).

Write \( \mathbb{I}^2 \) as a union of closed rectangles \( R \) so small that \( f(R) \subset A_\alpha \) for some \( \alpha \in \Lambda \). Arrange that the interiors of these rectangles are disjoint and that any vertex \( v \) of one of the rectangles lies in at most two other rectangles. (The rectangles are jiggled a bit to achieve this.) By suitably subdividing and reindexing
we may assume that each $f_i$ is (up to reparameterization) the restriction of $F$ to a union of consecutive intervals in $0 \times I$ each of which is the intersection of $0 \times I$ with one of these rectangles $R$.

Let $V$ be the set of all vertices of these rectangles. For each vertex $v \in V$ choose a path $g_v$ from $x_0$ to $F(v)$ lying in $A_\alpha(R_1) \cap A_\alpha(R_2) \cap A_\alpha(R_3)$ where $R_1, R_2, R_3$ are the rectangles which contain $v$. Call two vertices $u, v \in V$ adjacent if the line segment $[u, v]$ from $u$ to $v$ lies in an edge of one of the rectangles. For each pair $u, v$ of adjacent vertices let $e_{u,v} = F([u, v])$. Then the loop

$$f_{u,v} := g_u e_{u,v} g_v$$

lies in $A_\alpha \cap A_\beta$ where $R_1$ and $R_2$ are the two rectangles adjacent to $[u, v]$ and $f(R_1) \subset A_\alpha$ and $f(R_2) \subset A_\beta$. Denote the corresponding homotopy classes as

$$[f_{u,v}]_\alpha \in \pi_1(A_\alpha, x_0), \quad [f_{u,v}]_\beta \in \pi_1(A_\beta, x_0).$$

These are distinct elements of the free product $G$ but project to the same element of $G/N$.

Consider an element $h$ of the free product $G$ of form

$$h = [f_{v_0,v_1}]_\alpha [f_{v_1,v_2}]_\alpha \cdots [f_{v_{n-1},v_n}]_\alpha.$$

Replacing a subscript $\alpha_k$ by the another subscript $\beta_k$ as above changes the element $h \in G$ but leaves its image in $G/N$ unchanged. If two consecutive factors $[f_{v_{k-1},v_k}]_\alpha [f_{v_k,v_{k+1}}]_\alpha$ have the same subscript, i.e. if $\alpha_k = \alpha_{k+1} =: \alpha$, then $v_{k-1}, v_k, v_{k+1}$ are three of the four vertices of a rectangle $R$ and replacing $v_k$ by the fourth vertex $v_k'$ leaves $h$ unchanged (as an element of $G$). This each of these two operations (replacing $\alpha_k$ by $\beta_k$ and replacing $v_k$ by $v_k'$) leaves the image of $h$ in $G/N$ unchanged. But by means of these operations we can move the constant class in $G/N$ to the class of $f$ in $G/N$. (Represent $f$ as a product $h$ where the $v_j$ lie in $(0 \times I) \cup (I \times 1)$.) Thus a preimage of $[f]$ in $G$ lies in $N$ as claimed.

\[\square\]

**Example 4.4.** By Van Kampen, $\pi_1(X \lor Y) = \pi_1(X) \ast \pi_1(Y)$. In particular, the fundamental group of a wedge of circles is a free group with one generator for each circle.

**Example 4.5.** Any group $G$ is the fundamental group of of a two dimensional cell complex $X$ as follows. Represent $G$ as a quotient $G = F/H$ where $F$ is the free group on the generators $\{x_i\}_{i \in I}$ and $H$ is the normal subgroup generated by the words $\{r_\alpha\}_{\alpha \in \Lambda}$. Assume that the 1-skeleton $X^{(1)}$ of $X$ is a wedge of circles $X^{(1)} := \bigvee_{i \in I} S^1$ with one circle for each generator $x_i$. Form $X$ by attaching each a 2-cell $e^2_\alpha$ for each relation $r_\alpha$ via a map $\phi_\alpha : S^1 \to X^1$ representing $r_\alpha \in F = \pi_1(X^{(1)})$. For each $\alpha \in \Lambda$ choose $z_\alpha \in e^2_\alpha$. Let $U$ be a contractible neighborhood of the wedge point not containing any $z_\alpha$ and

$$A = X \setminus \{z_\alpha : \alpha \in \Lambda\}, \quad B = U \cup (X \setminus X^{(1)}).$$
Then \( A \) and \( B \) are open sets, \( A \) deformation retracts onto \( X^{(2)} \), \( B \) is contractible, and \( A \cap B \) is homotopy equivalent to a wedge of circles with one circle for each relation \( r_\alpha \). Since \( \pi_1(B) \) is trivial the free product \( \pi_1(A) \ast \pi_1(B) \) is \( \pi_1(A) \) and theVan Kampen subgroup \( N \) is the normal subgroup generated by \( \{r_\alpha\}_{\alpha \in A} \).

**Remark 4.6.** Shelah [8] proves that the fundamental group of a compact metric space is either finitely generated or uncountable.

**4.7.** A **link** is the image \( K = f(S) \) of an embedding \( f : S \to \mathbb{R}^3 \) of a finite disjoint union of circles. A link with one component (i.e. \( S = S^1 \)) is called a **knot**. The fundamental group \( \pi_1(\mathbb{R}^3 \setminus K) \) is called the group of the link or knot. One generally assumes that the embedding is smooth or piecewise linear (the precise terminology is *tame*) to avoid pathologies. There is an algorithm which computes the group of a knot or more precisely produces a presentation of the group known as the **Wirtinger representation**. It is described on page 55 of [3].

**Exercise 4.8.** Use the Wirtinger representation to prove that the Abelianization of the group of a knot is \( \mathbb{Z} \).

**Example 4.9.** The complement \( \mathbb{R}^3 \setminus S^1 \) of the unknot deformation retracts onto a space homeomorphic to \( S^1 \vee S^2 \) so \( \pi_1(\mathbb{R}^3 \setminus S^1) = \mathbb{Z} \).

**Exercise 4.10.** Show that if \( \mathbb{R}^3 \setminus K \simeq X \) then \( \mathbb{R}^3 \setminus K \simeq S^2 \vee X \). Hence \( \pi_1(\mathbb{S}^3 \setminus K) = \pi_1(\mathbb{R}^3 \setminus K) \).

**Example 4.11.** The complement \( \mathbb{S}^3 \setminus L \) of two linked circles

\[
L = S^1 \times 0 \cup 0 \times S^1 \subset S^3 \subset \mathbb{C}^2
\]

is homeomorphic to \( T^2 \times \mathbb{R} \) via the map

\[
\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R} \to L : (e^{i\alpha}, e^{i\beta}, t) \mapsto \left( \frac{e^{t+i\alpha}}{\sqrt{e^{2t} + e^{-2t}}}, \frac{e^{-t+i\beta}}{\sqrt{e^{2t} + e^{-2t}}} \right)
\]

so (under a suitable embedding) \( \pi_1(\mathbb{R}^3 \setminus L) = \pi_1(\mathbb{S}^2 \vee T^2) = \pi_1(T^2) = \mathbb{Z}^2 \).

A suitable stereographic projection sends \( L \) to the disjoint union of the circle \( x^2 + y^2 = 1, z = 0 \) with the \( z \)-axis.

**Example 4.12.** The complement \( \mathbb{S}^3 \setminus L' \) of two unlinked circles deformation retracts \( X \simeq S^1 \vee S^2 \vee S^1 \vee S^2 \) so \( \pi_1(\mathbb{R}^3 \setminus L') \) is a free group on two generators.

**Example 4.13.** Let \( p, q \in \mathbb{Z} \). The **torus knot** of type \( (p,q) \) is the knot

\[
K := \{ (u,v) \in \mathbb{S}^1 \times \mathbb{S}^1 : u^p = v^q \} \subset \mathbb{S}^3(\sqrt{2})
\]

where \( \mathbb{S}^3(\sqrt{2}) := \{ (u,v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 2 \} \) denotes the sphere of radius \( \sqrt{2} \).

**Exercise 4.14.** The one point compactification of \( \mathbb{R}^3 \) is (homeomorphic to) \( \mathbb{S}^3 \).

Show that if \( K \subset \mathbb{R}^3 \) is compact and \( \mathbb{S}^3 \setminus K \simeq Y \), then \( \mathbb{R}^3 \setminus K \simeq S^2 \vee Y \).
Exercise 4.15. Let $Y$ result from the cube $I^3$ by identifying opposite faces with a $90^\circ$ right hand twist. Use Van Kampen’s Theorem to show that the fundamental group of $Y$ is the eight element quaternion group

$$G = \{ \pm 1, \pm i, \pm j, \pm k \}.$$  

Then give another proof by showing that the universal cover is $S^3 \to S^3/G \cong Y$. Hint: $S^3 \cong \partial [-1,1]^4$ and each face of $[-1,1]^4$ is a cube.

Exercise 4.16. Conley [1] page 26 studies a flow entering a cylinder $X = \mathbb{D}^2 \times I$ at the top $\mathbb{D}^2 \times 1$, exiting at the bottom $\mathbb{D}^2 \times 0$, with the orbits running vertically downward on the side $S^1 \times I \subset \partial X$, and with a knotted orbit segment $A$ inside the cylinder running from the point $p = (0,0,1) \in \mathbb{D}^2 \times 1$ in the top of the cylinder to a point $q = (0,0,0) \in \mathbb{D}^2 \times 0$ in bottom. He argues that there must be a nonempty invariant set inside the cylinder since otherwise the complement of the knotted orbit in the cylinder would deformation retract onto the punctured disk at the bottom. Prove that $(\mathbb{D}^2 \times 0) \setminus q$ is not a deformation retract of $X \setminus A$. In this context the condition that $A$ is knotted means that $\pi_1(X \setminus A)$ is not $\mathbb{Z}$. To see that this can occur, show that $\pi_1(X \setminus A) = \pi_1(\mathbb{R}^3 \setminus K)$ where $K$ is constructed from $A$ by adjoining the polygonal arc with successive vertices $p_0 = 0$, $p_1 = (0,0,2)$, $p_2 = (0,2,2)$, $p_3 = (0,2,-2)$, $p_4 = (0,0,-2)$, and $p_5 = q$.

References
