Maxwell’s Equations

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Let \((t, x, y, z)\) be a coordinate system on \(\mathbb{R}^4\). The following are defined in physics.

\[
\begin{align*}
E &= E_x \frac{\partial}{\partial x} + E_y \frac{\partial}{\partial y} + E_z \frac{\partial}{\partial z} \\
B &= B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \\
D &= D_x \frac{\partial}{\partial x} + D_y \frac{\partial}{\partial y} + D_z \frac{\partial}{\partial z} \\
H &= H_x \frac{\partial}{\partial x} + H_y \frac{\partial}{\partial y} + H_z \frac{\partial}{\partial z} \\
j &= j_x \frac{\partial}{\partial x} + j_y \frac{\partial}{\partial y} + j_z \frac{\partial}{\partial z} \\
\rho &= \rho(t, x, y, z)
\end{align*}
\]

(electric field) (magnetic induction) (electric displacement) (magnetic intensity) (current density) (charge density)

\[
\begin{align*}
D_x &= \varepsilon_0 E_x, \quad D_y = \varepsilon_0 E_y, \quad D_z = \varepsilon_0 E_z, \\
B_x &= \mu_0 H_x, \quad B_y = \mu_0 H_y, \quad B_z = \mu_0 H_z,
\end{align*}
\]

(vacuum constitutive equations)

where \(\varepsilon_0\) and \(\mu_0\) are constants.
Define

\[ F = -E_x \, dt \, dx - E_y \, dt \, dy - E_z \, dt \, dz \]
\[ + B_x \, dy \, dz + B_y \, dz \, dx + B_z \, dx \, dy, \]

\[ K = H_x \, dt \, dx + H_y \, dt \, dy + H_z \, dt \, dz \]
\[ + D_x \, dy \, dz + D_y \, dz \, dx + D_z \, dx \, dy, \]

\[ J = \rho \, dx \, dy \, dz - j_x \, dt \, dy \, dz \]
\[ - j_y \, dt \, dz \, dx - j_z \, dt \, dx \, dy. \]

(I) Show that

\[ dF = 0 \iff \begin{cases} \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \cdot B = 0 \end{cases} \]

\[ dK = J \iff \begin{cases} \nabla \times H = j + \frac{\partial D}{\partial t} \\ \nabla \cdot D = \rho \end{cases} \]

\[ dJ = 0 \iff \frac{\partial \rho}{\partial t} + \nabla \cdot j = 0. \]

The equations \( dF = 0 \) and \( dK = J \) are called Maxwell’s equations. The equation \( dJ = 0 \) is called the law of conservation of charge. Since \( d^2 = 0 \), conservation of charge is a consequence of Maxwell’s equations.

Let \((t', x', y', z')\) be a new coordinate system on \(\mathbb{R}^4\) and define functions \(E_{x'}, \ldots, H_{z'}\) by

\[ F = -E_{x'} \, dt' \, dx' - E_{y'} \, dt' \, dy' - E_{z'} \, dt' \, dz' \]
\[ + B_{x'} \, dy' \, dz' + B_{y'} \, dz' \, dx' + B_{z'} \, dx' \, dy', \]

\[ K = H_{x'} \, dt' \, dx' + H_{y'} \, dt' \, dy' + H_{z'} \, dt' \, dz' \]
\[ + D_{x'} \, dy' \, dz' + D_{y'} \, dz' \, dx' + D_{z'} \, dx' \, dy'. \]
(II) Suppose that \( v \) and \( c \) are constants and that

\[
t = \gamma \left( t' + \frac{vx'}{c^2} \right), \quad x = \gamma (x' + vt'), \quad y = y', \quad z = z',
\]

where

\[
\gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-1/2}.
\]

Assume that

\[
c^{-2} = \varepsilon_0\mu_0.
\]

(1) Compute \( E_{x'}, \ldots, B_{z'} \), in terms of \( E_x, \ldots, B_z \).

(2) Compute \( D_{x'}, \ldots, H_{z'} \), in terms of \( E_x, \ldots, B_z \).

(3) Give an example where \( E_x, E_y, E_z \) vanish identically, but \( E_{x'}, E_{y'}, E_{z'} \) do not.

(4) Give an example where \( B_x, B_y, B_z \) vanish identically, but \( B_{x'}, B_{y'}, B_{z'} \) do not.

(5) Show that if \( E_x, \ldots, H_z \) satisfy the vacuum constitutive equations, then so do \( E_{x'}, \ldots, H_{z'} \).

(III) Suppose that \( v \) is a constant and that

\[
t = t', \quad x = x' + vt', \quad y = y', \quad z = z'.
\]

(1) Compute \( E_{x'}, \ldots, B_{z'} \), in terms of \( E_x, \ldots, B_z \).

(2) Compute \( D_{x'}, \ldots, H_{z'} \), in terms of \( E_x, \ldots, B_z \).

(3) Give an example where \( E_x, E_y, E_z \) vanish identically, but \( E_{x'}, E_{y'}, E_{z'} \) do not.

(4) Show that if \( B_x, B_y, B_z \) vanish identically, then so do \( B_{x'}, B_{y'}, B_{z'} \).

(5) Give an example where \( E_x, \ldots, H_z \) satisfy the vacuum constitutive equations, but \( E_{x'}, \ldots, H_{z'} \) do not (even if the constants \( \epsilon_0 \) and \( \mu_0 \) are allowed to change).
(IV) Suppose that \( \Omega \subset \{0\} \times \mathbb{R}^3 \) is a compact 3-manifold with boundary \( \Sigma = \partial \Omega \). Assume \( dJ = 0 \). Show that
\[
\frac{d}{dt} \int_\Omega \rho \, dx \, dy \, dz = - \int_\Sigma \iota(j) \, dx \, dy \, dz.
\]
where \( \iota(j) \, dx \, dy \, dz = j_x \, dy \, dz + j_y \, dz \, dx + j_z \, dx \, dy \).

(V) Let \( r = \sqrt{x^2 + y^2 + z^2} \) and assume that
\[
F = d \left( \frac{dt}{r} \right).
\]
This \( F \) is only defined on \( M = \mathbb{R}^4 \setminus (\mathbb{R} \times \{(0,0,0,0)\}) \). Show that if \( K \) is determined by the constitutive equations, then
\[
K = k \frac{xdydz + ydzdx + zdxdy}{r^3}
\]
for a suitable constant \( k \). Show that
\[
\int_\Sigma K = 4\pi k
\]
for any \( \Omega \) as in (IV) having the origin in its interior. Conclude that \( K \) is closed but not exact on \( M \).