Exercise 1. Let $X$ and $Y$ be open in $\mathbb{R}^n$ and $f : X \to Y$ be a smooth map, then

$$f^*(dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n) = \det(DF(x))dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$ 

Here $x_1, \ldots, x_n$ are coordinates on $X$ and $y_1, \ldots, y_n$ are coordinates on $Y$ and $DF(x)$ denotes the matrix of partial derivatives.

Exercise 2. Let $f : (0, \infty) \times (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ be the spherical coordinates map $f(\rho, \theta, \phi) = (x, y, z)$ where

$$x = \rho \cos \phi \cos \theta, \quad y = \rho \cos \phi \sin \theta, \quad z = \rho \sin \phi.$$ 

Compute $f^*(dx \wedge dy \wedge dz)$ and $f^*(x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy)$.

Exercise 3. Let $h$ be a function and $v = (v_1, v_2, v_3)$ be a vector field both defined on an open subset of $\mathbb{R}^3$. Define the gradient vector field $\nabla h$, the curl $\nabla \times v$, and the divergence $\nabla \cdot v$ as in vector calculus and let

$$\omega = v_1 \, dy \wedge dz + v_2 \, dz \wedge dx + v_3 \, dx \wedge dy, \quad \lambda = v_1 \, dx + v_2 \, dy + v_3 \, dz.$$ 

Compare the components of the 1-form $dh$ and the gradient vector field $\nabla h$, of the 2-form $d\lambda$ and the curl $\nabla \times v$, and of the 3-form $d\omega$ and the divergence $\nabla \cdot v$. What is $\iota(v)(dx \wedge dy \wedge dz)$? What is $\iota(v)\lambda$? What is $\iota(v)\omega$?

Exercise 4. Let

$$O_3 := \{ \Phi \in \text{GL}_3(\mathbb{R}) : \Phi^* = \Phi^{-1} \}$$

denote the group of all linear transformations $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ which preserve distance and

$$SO_3 := \{ \Phi \in O_3 : \det(\Phi) = 1 \}$$

denote the subgroup of those transformations which preserve orientation as well. Define $\omega \in \Omega^2(\mathbb{R}^3)$ by

$$\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$$

where $x, y, z$ are the usual coordinates on $\mathbb{R}^3$. Let $\rho := \sqrt{x^2 + y^2 + z^2}$ be the distance from the origin and let $\eta \in \Omega^2(\mathbb{R}^3 \setminus 0)$ be a two form defined in the complement of the origin. Show that
Exercise 6. Interpret and then prove the formula

\[ 2\omega = (r \times dr) \cdot dr \]

where \( r = (x, y, z) \), \( dr = (dx, dy, dz) \) and the dot product \( u \cdot v \) and cross product \( u \times v \) are as in calculus.

Exercise 7. According to Ampère’s law the magnetic field at a point \( p \) due to the current at \( q \) in the segment \( dq \) is

\[ dB = -\frac{1}{4\pi} \left( \frac{p - q}{|p - q|^3} \times dq \right) \]
so that total magnetic field at $p$ due to the current in a closed loop $\Gamma$ is

$$B(p) = -\frac{1}{4\pi} \int_{q \in \Gamma} \frac{(p - q) \times dq}{|p - q|^3}.$$ 

Show that the corresponding one form $\beta$ is closed and generates $H^1_{DR}(\mathbb{R}^3 \setminus \Gamma)$. Hint: Show (using the form $\rho^{-3}\omega$ of Exercise 4) that it is enough to find a smooth loop $\Lambda$ in $\mathbb{R}^3 \setminus \Gamma$ such that the degree of the map $f : \Lambda \times \Gamma \to S^2$ given by

$$f(p, q) = \frac{p - q}{|p - q|}$$

is not zero. Assume that the $z$ coordinate on $\Gamma$ is maximized at the point $p_0 = (x_0, y_0, z_0)$, choose $\Lambda$ to be a tiny circle centered at $p_0$ in the orthogonal complement to the tangent line to $\Gamma$ at $p_0$, and show that $(0, 0, 1)$ is a regular value of $f$.

**Exercise 8.** Show that the inclusion map of smooth singular chains into (continuous) singular chains induces an isomorphism between singular homology and smooth singular homology.