Exercise 1. Suppose that \( n \) and \( m \) are integers with \( n > m > 0 \). Let \( M^m \) and \( N^n \) denote compact, connected manifolds without boundary of dimensions \( m \) and \( n \) respectively. Prove that they are not homotopy equivalent.

Exercise 2. Give an example of a pair of path connected spaces \( X_1, X_2 \) such that \( \pi_1(X_1) \cong \pi_1(X_2) \neq 1 \), but \( H_*(X_1, \mathbb{Z}) \neq H_*(X_2, \mathbb{Z}) \).

Exercise 3. Let \( M = \mathbb{C}P^n \) denote complex projective space of (real) dimension \( 2n \).

(a) Prove that \( M \) is a compact, connected \( 2n \)-dimensional manifold.

(b) Show in detail that \( M \) can be given a cell complex structure with one cell in every even dimension \( i = 0, 2, 4, \ldots, 2n \).

(c) Calculate the cohomology ring \( H^*(M, \mathbb{Z}) \).

(d) Find a closed \((2n - 2k)\)-form \( \eta_S \) representing the Poincaré dual of the submanifold \( S := \mathbb{C}P^k \subset \mathbb{C}P^n \) in \( H^{2n-2k}_{DR}(M) \).

Exercise 4. Let \( M = \mathbb{R}P^n \) denote real projective space of (real) dimension \( n \).

(a) Prove that \( M \) is a compact, connected \( n \)-dimensional manifold.

(b) Show in detail that \( M \) can be given a cell complex structure with one cell in every even dimension \( i = 0, 1, 2, \ldots, n \).

(c) Calculate the cohomology ring \( H^*(M, \mathbb{Z}/2) \)

(d) Calculate the cohomology ring \( H^*(M, \mathbb{Z}) \).

Exercise 5. Let \( M \) denote a connected, non-orientable, compact 3–manifold without boundary. Prove that its fundamental group must be an infinite group.

Exercise 6. Let \( X \) be a finite cell complex and \( SX \) its suspension, i.e.

\[
SX := X \times \mathbb{I}/\sim
\]

where \( (x, 0) \sim (x', 0) \) and \( (x, 1) \sim (x', 1) \) for \( x, x' \in X \). Prove that that the reduced cohomology \( \tilde{H}^*(SX, \mathbb{Z}) \) has no non-trivial cup products.
Exercise 7. Let $S^{2n-1} \subset C^n$ and $\rho = e^{2\pi i/p}$ (p an odd prime). Let $Z/p$ act on $S^{2n-1}$ by $x \mapsto \rho x$.

(a) Show that this action is free.

(b) Calculate $H_*(S^{2n-1}/Z/p, Z)$.

(c) Calculate $H_*(S^{2n-1}/Z/p, Z/p)$.

Exercise 8. Let $X$ be a path–connected finite CW complex and let $S^1$ denote the unit circle. Show that $[X, S^1]$, the set of all homotopy classes of maps $f : X \to S^1$, has a natural group structure induced from the product on $S^1$. Prove that there is an isomorphism of groups $[X, S^1] \cong H^1(X, Z)$.

Exercise 9. Calculate $H_*(RP^3 \times (S^5/Z/3) \times S^2, Z)$.

Exercise 10. Using cup products show that if $m > n$ there is no map $RP^m \to RP^n$ inducing a non trivial map $H^1(RP^n, Z/2) \to H^1(RP^m, Z/2)$. Derive the Borsuk Ulam Theorem as a consequence.

Exercise 11. Show that $RP^3$ and $RP^2 \vee S^3$ are not homotopy equivalent.