1. A graded vector space is a sequence $C = \{C_k\}_k$ of vector space $C_k$ indexed by the integers. We also denote by $C$ the direct sum

$$C = \bigoplus_k C_k.$$  

A graded map of degree $s$ between two graded vector spaces $C$ and $C'$ is a linear map $\psi : C \to C'$ such that

$$\psi(C_k) \subset C'_{k+s};$$

equivalently, it is a sequence of linear maps $\psi_k : C_k \to C'_{k+s}$. A chain complex is a pair $(C, \partial)$ consisting of a graded vector space and a graded linear map $\partial : C \to C$ of degree $-1$ such that $\partial^2 = 0$, i.e.

$$\text{image}(\partial : C_{k+1} \to C_k) \subset \ker(\partial : C_k \to C_{k-1}).$$

The homology $H(C, \partial)$ of the complex is the quotient of the kernel of $\partial$ by the image of $\partial$. It inherits a grading

$$H_k(C, \partial) = \frac{\ker(\partial : C_k \to C_{k-1})}{\text{image}(\partial : C_{k+1} \to C_k)}.$$  

Elements of the kernel (resp. image) of $\partial$ are called cycles (resp. boundaries) and the elements of $H(C, \partial)$ are called homology classes. We will denote the homology class of a cycle $c$ by $[c]$:  

$$[c] = \{c + \partial b : b \in C_{k-1}\}, \quad c \in C_k, \quad \partial c = 0.$$  

A cochain complex is the same as a chain complex except that the differential $\partial$ has degree $+1$; cohomology, cocycles, coboundaries, etc. are
defined analogously. Thus the dual $\partial^* : C^* \to C^*$ of a chain complex is a cochain complex and the dual of a cochain complex is a chain complex. It is customary to use subscripts with chain complexes and superscripts with cochain complexes. We sometimes write $C_*$ instead of $C$ to emphasize the grading and

$$\psi : C_* \to C_{s+s}$$

to indicate that $\psi$ is a graded map of degree $s$.

**Example 2.** The sequence

$$\cdots \to \mathcal{E}^{k-1}(M) \xrightarrow{d} \mathcal{E}^k(M) \xrightarrow{d} \mathcal{E}^{k+1}(M) \to \cdots$$

where $\mathcal{E}^k(M)$ is the vector space of differential $k$-forms on a smooth manifold $M$ and $d : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$ the exterior derivative is a cochain complex. The corresponding cohomology $H^k(M)$ is called the **de Rham cohomology** of $M$. Thus a closed form is a cocycle and an exact form is a coboundary.

**Example 3.** Replacing $\mathcal{E}^k(M)$ by the subspace $\mathcal{E}^k_c(M)$ of forms of compact support gives a sub chain complex whose cohomology is called **compactly supported de Rham cohomology** and is denoted $H^k_c(M)$.

**Example 4.** Let $M$ be a smooth manifold and $S_k(M)$ denote the space of **singular $k$-chains** on $M$ defined as follows. (A **singular $k$-simplex** is a smooth map $\sigma : \Delta^k \to M$ where

$$\Delta^k = \left\{ (t_0, t_1, \ldots, t_k) \in [0, 1]^{k+1} : \sum_{j=0}^k t_j = 1 \right\}$$

is the **standard $k$-simplex**; a **singular $k$-chain** is a finite formal sum

$$c = \sum_{i=1}^n c_i \sigma_i$$

where each $\sigma_i$ is a singular $k$-simplex and the coefficients $c_i$ are real numbers.) For $j = 0, 1, \ldots, k$ define the $j$th **face map** $\iota_j : \Delta^{k-1} \to \Delta^k$ by

$$\iota_j(t_1, \ldots, t_k) = (t_1, \ldots, t_j, 0, t_{j+1}, \ldots, t_k).$$

The boundary operator $\partial : S_k(M) \to S_{k-1}(M)$ is linear map such that

$$\partial \sigma = \sum_{j=0}^k (-1)^j \sigma \circ \iota_j$$
for a singular $k$-simplex $\sigma$. It is easy to check that $\partial^2 \sigma = 0$ so that $(S_*(M), \partial)$ forms a chain complex. The corresponding homology $H_* (M)$ is called the **singular homology** of $M$. The cohomology of the dual complex

$$S^k(M) = S_k(M)^*$$

is called **singular cohomology** and denoted $H^*_k(M)$. A singular cochain $\alpha \in S^k(M)$ is said to have **compact support** iff there is a compact set $K \subset M$ such that $\alpha(\sigma) = 0$ whenever $\sigma : \Delta^k \to M \setminus K$. We denote the sub complex of $S^*(M)$ consisting of the cochains of compact support by $S^*_{cs}(M)$. The cohomology $S^*_{cs}(M)$ is denoted $H^*_{cs}(M)$ and called **compactly supported singular cohomology**.

5. **A chain map** from the complex $(C, \partial)$ to the complex $(C', \partial')$ is a graded linear map of degree zero such that $\partial' \circ \psi = \psi \circ \partial$. It follows that $\psi$ maps cycles to cycles and boundaries to boundaries and so induces a map

$$H_k(\psi) : H_k(C_*, \partial) \to H_k(C'_*, \partial')$$

on homology. (We may also denote the induced map by $\psi_*$.)

**Example 6.** The inclusion $E^k_c(M) \to E^k(M)$ defined by extension by zero is a chain map and induces a map

$$H^k_c(M) \to H^k(M).$$

**Example 7.** Each smooth map $f : M \to N$ between smooth manifolds yields a chain map $f^* : E^k(N) \to E^k(M)$ and hence an induced map

$$H^k(f) : H^k(N) \to H^k(M)$$

on de Rham cohomology. In particular, the inclusion of an open set $U$ into its ambient space $M$ induces a restriction map $E^k(M) \to E^k(U)$.

**Example 8.** If $f : M \to N$ and $\omega \in E^k_c(N)$ then $f^* \omega \in E^k_c(M)$ provided that $f$ is proper, i.e. the preimage $f^{-1}(K)$ of a compact subset $K$ of $N$ is a compact subset of $M$. Note that the inclusion map of an open subset into its ambient space is usually not proper.

**Example 9.** A smooth map $f : M \to N$ induces a chain map $f_* : S_k(M) \to S_k(N)$ on singular homology via the formula

$$f_* \sigma = f \circ \sigma$$

(extend by linearity) for each singular $k$-simplex $\sigma : \Delta^k \to M$.  

**Example 10.** Given a singular chain $c \in S_k(M)$ and a smooth form $\omega \in \mathcal{E}^k(M)$ define the integral of $\omega$ along $c$ by

$$\int_c \omega = \sum_{i=1}^n c_i \int_{\Delta_i} \sigma_i^* \omega$$

with $c$ as in paragraph 4. This defines a map

$$\mathcal{E}^k(M) \to S^k(M) := S_k(M)^* : \omega \mapsto \left( c \mapsto \int_c \omega \right).$$

By Stokes’ formula

$$\int_c d\omega = \int_{\partial c} \omega$$

this map $\mathcal{E}^k(M) \to S_k(M)^*$ is a chain map from $(\mathcal{E}^*(M), d)$ to $(S_*(M)^*, \partial^*)$.

It is easy to see that for any smooth map $f : M \to N$ the diagram

$$\begin{array}{ccc}
\mathcal{E}^k(N) & \xrightarrow{f^*} & \mathcal{E}^k(N)(M) \\
\downarrow & & \downarrow \\
S^k(N) & \xrightarrow{(f_*)^*} & S^k(M)
\end{array}$$

commutes.

**11.** Two chain maps $\psi_0 : (C_*, \partial) \to (C'_*, \partial')$ and $\psi_1 : (C_*, \partial) \to (C'_*, \partial')$ are called **chain homotopic** iff there is a sequence of linear maps $\Psi : C_* \to C'_{*+1}$ such that

$$\psi_1 - \psi_0 = \partial' \circ \Psi + \Psi \circ \partial.$$

It is easy to see that chain homotopic chain maps induce the same map on homology:

$$H_k(f_0) = H_k(f).$$

**12.** Two smooth maps $f_0 : M \to N$ and $f_1 : M \to N$ are called **homotopic** iff there is a smooth map $F : I \times M \to N$, $I = [0, 1]$ such that $F(t, \cdot) = f_t$ for $t = 0, 1$.

**Theorem 13.** If $f_0 : M \to N$ and $f_1 : M \to N$ are homotopic, then the induced chain maps $f_0^* : \mathcal{E}^*(N) \to \mathcal{E}^*(M)$ and $f_1^* : \mathcal{E}^*(N) \to \mathcal{E}^*(M)$ are chain homotopic.
Proof. Denote Lie differentiation by $\ell$ and interior multiplication by $\iota$. Let $T$ be the vector field on $I \times M$ defined by

$$\ell(T) = \frac{\partial}{\partial t}$$

where $t$ is the coordinate on $I$. Cartan’s formula is

$$\ell(T) = \iota(T)d + d\iota(T).$$

For $t \in I$ define $j_t : M \to I \times M$ by

$$j_t(p) = (t, p).$$

Let $F : I \times M \to N$ between $f_0$ and $f_1$. Define $\Phi : \mathcal{E}^k(N) \to \mathcal{E}^{k-1}(M)$ by

$$\Phi(\omega) = \int_0^1 j_t^* \ell(T) F^* \omega \, dt.$$  

By Cartan’s formula

$$j_t^* \ell(T) F^* \omega = j_t^* \iota(T) F^* (d \omega) + d(j_t^* \iota(T) F^* \omega)$$

for $\omega \in \mathcal{E}^k(N)$. Now integrating over $I$ gives the homotopy formula

$$f_1^* \omega - f_0^* \omega = \int_0^1 j_t^* \ell(T) \omega \, dt = \Phi(d \omega) + d\Phi(\omega).$$

The first equality is an application of Fundamental Theorem of Calculus.

14. Two chain complexes $(C, \partial)$ and $(C', \partial')$ are called chain equivalent iff there are chain maps $\psi : C \to C'$ and $\phi : C' \to C$ such that $\phi \circ \psi$ is chain homotopic to the identity map of $C$ and $\psi \circ \phi$ is chain homotopic to the identity map of $C'$. It follows that $H((\psi))$ is an isomorphism whose inverse is $H(\phi)$. Two manifolds $M$ and $N$ are homotopy equivalent if there are smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ is homotopic to the identity map of $M$ and $f \circ g$ is homotopic to the identity map of $N$.

Corollary 15 (Poincaré Lemma). Assume that the manifold $M$ is contractible, i.e. that the identity map of $M$ is homotopic to a constant map. Then $H^0(M) = \mathbb{R}$ and $H^k(M) = 0$ for $k > 0$. 

5
Proof. A manifold is contractible if and only if it is homotopy equivalent to a point.

**Corollary 16.** Let $M$ be a compact manifold and $p \in M$. Then the chain map

$$\mathcal{E}_c^*(M \setminus p) \to \mathcal{E}^*(M)$$

given by extension by zero induces an isomorphism $H_c^k(M \setminus p) \simeq H^k(M)$ for $k > 0$.

**Proof.** Choose a neighborhood of $p$ which is diffeomorphic to a ball and use it to construct a smooth homotopy $f_t : M \to M$ for $0 \leq t \leq 1$ such that $f_t(q) = q$ for $d(q,p) \geq 2t$, $f_t(q) = p$ for $d(q,p) \leq t$, and $d(f_t(q), p) \leq d(q, p)$ for all $t$ and $q$. (Here $d(q, p)$ is the distance from $p$ to $q$ if $q$ is in the neighborhood and is infinite otherwise.)

We show that the map $H_c^k(M \setminus p) \to H^k(M)$ is surjective. Choose $\omega \in \mathcal{E}^k(M)$ with $d\omega = 0$. Then there is a $\theta \in \mathcal{E}^k(M)$ with $\omega = f_1^* \omega + d\theta$ since $f_1$ is homotopic to the identity. But $f_1^* \omega \in \mathcal{E}_c^k(M \setminus p)$ since $f_1$ is a constant map in a neighborhood of $p$. (Note that the argument fails for $k = 0$.)

We show that the map $H_c^k(M \setminus p) \to H^k(M)$ is injective. Choose $\omega_c \in \mathcal{E}_c^k(M \setminus p)$ with $\omega_c = d\theta$ with $\theta \in \mathcal{E}^{k-1}(M)$. Then $\omega_c = f_t^* \omega_c = df_t^* \theta$ for $t$ sufficiently small. If $k > 1$ then $f_t^* \theta \in \mathcal{E}_c^{k-1}(M \setminus p)$; if $k = 1$ then $f_t^* \omega$ is a constant $c$ near $p$ and $f_t^* \theta - c \in \mathcal{E}_c^{k-1}(M \setminus p)$. In either case we have found a $\theta_c \in \mathcal{E}_c^{k-1}(M \setminus p)$ with $\omega_c = d\theta_c$. \qed

**Remark 17.** A closed zero form is locally constant so if $M$ is connected we have

$$H^0(M) = \mathbb{R}, \quad H_c^0(M) = \mathbb{R}.$$ 

It is easy to see that the (co)homology of a disconnected manifold is the direct sum of the (co)homology of its connected components (at least this is easy for the theories introduced above) so we will usually assume that $M$ is connected.

**18.** A sequence

$$\cdots \to C_{k+1} \to C_k \to C_{k-1} \to \cdots$$

of vector spaces and linear maps between is called **exact** iff the image of each map is equal to the kernel of the next. Equivalently the sequence is exact iff
it is an acyclic chain complex, i.e. a chain complex whose homology is zero. An exact sequence of form

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$$

is called a short exact sequence. For a short exact sequence, the vector space $C$ is isomorphic to the quotient of $B$ by the image of $A$. The five lemma says that if the rows of the commutative diagram

$$\begin{array}{cccccc}
C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
C'_1 & \rightarrow & C'_2 & \rightarrow & C'_3 & \rightarrow \\
\end{array}$$

are exact and the first, second, fourth, and fifth vertical arrows are isomorphisms, then the third vertical arrow is also an isomorphism. The proofs of this and of Lemma 19 and Theorem 20 which follow may be found in any introductory text in algebraic topology.

**Lemma 19.** Consider a short exact sequence

$$0 \rightarrow (A, \partial_A) \xrightarrow{i} (B, \partial_B) \xrightarrow{j} (C, \partial_C) \rightarrow 0$$

of chain maps between chain complexes. Then there is a unique graded linear map

$$\partial_H : H_*(C, \partial_C) \rightarrow H_{*-1}(A, \partial_A)$$

such that

$$\partial_H[c] = [\partial_Aa]$$

whenever $a \in A, b \in B, c \in C$ satisfy $i(a) = \partial_B(b)$ and $j(b) = c$. (The last two conditions imply that $\partial_A(a) = 0$ and $\partial_C(c) = 0$.)

**Theorem 20 (Long Exact Sequence).** In the situation of Lemma 19 the sequence

$$\cdots \xrightarrow{\partial_H} H_k(A, \partial_A) \xrightarrow{i_*} H_k(B, \partial_B) \xrightarrow{j_*} H_k(C, \partial_C) \xrightarrow{\partial_H} H_{k-1}(A, \partial_A) \xrightarrow{i_*} \cdots$$

is exact.

**Example 21.** Let $U$ and $V$ be open subsets of a manifold $M$. Then the sequence

$$0 \rightarrow \mathcal{E}^*(U \cup V) \rightarrow \mathcal{E}^*(U) \oplus \mathcal{E}^*(V) \rightarrow \mathcal{E}^*(U \cap V) \rightarrow 0$$
is exact. Here the map $E^*(U \cup V) \to E^*(U) \oplus E^*(V)$ sends $\gamma$ to $(\gamma|_U, \gamma|_V)$ and the map $E^*(U) \oplus E^*(V) \to E^*(U \cap V)$ sends $(\alpha, \beta)$ to $(\alpha|_{U \cap V} - (\beta|_{U \cap V})$. To see that the later map is surjective use a partition of unity subordinate to the cover $\{U, V\}$ of $U \cup V$: if $\text{supp}(\kappa) \subset U$ and $\omega$ is defined on $U \cap V$ then $\kappa\omega$ extends (by zero) to $V$. The resulting long exact sequence

$$\cdots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(U \cup V) \to \cdots$$

is called the **Mayer Vietoris sequence** in de Rham cohomology.

**Example 22.** In the situation of paragraph 21 the sequence

$$0 \to E^*_c(U \cap V) \to E^*_c(U) \oplus E^*_c(V) \to E^*_c(U \cup V) \to 0$$

is also exact. Here the map $E^*_c(U \cup V) \to E^*_c(U) \oplus E^*_c(V)$ sends $\gamma$ to $(\gamma, \gamma)$ extended by zero and the map $E^*_c(U) \oplus E^*_c(V) \to E^*_c(U \cap V)$ sends $(\alpha, \beta)$ to $\alpha - \beta$ extended by zero. Once again use partitions to show that the latter map is surjective: if $\text{supp}(\kappa)$ and $\omega \in E^*_c(U \cup V)$ then $\kappa\omega \in E^*_c(U)$. The resulting long exact sequence

$$\cdots \to H^k_c(U \cap V) \to H^k_c(U) \oplus H^k_c(V) \to H^k_c(U \cup V) \to H^{k+1}_c(U \cap V) \to \cdots$$

is called the **Mayer Vietoris sequence** in compactly supported de Rham cohomology.

**23.** The sequence

$$0 \to S_k(U \cap V) \to S_k(U) \oplus S_k(V) \to S_k(U \cup V) \to 0$$

is *not* exact; a singular simplex can lie in $U \cap V$ without lying in either $U$ or $V$. One can overcome this difficulty by subdivision and still define (long exact) Mayer Vietoris sequences in singular homology, singular cohomology, and compactly supported singular cohomology.

**Example 24.** The $n$-sphere $S^n$ is the union of two open sets $U$ and $V$, the complements of the north and south poles respectively, each diffeomorphic to $\mathbb{R}^n$. The intersection $U \cap V$ is homotopy equivalent to $S^{n-1}$. By induction on $n$ and the Mayer Vietoris sequence it follows that $H^k(S^n) = \mathbb{R}$ for $k = 0, n$ and vanishes otherwise. From Corollary 16 it follows that $H^k_c(\mathbb{R}^n) = \mathbb{R}$ for $k = n$ and vanishes otherwise.
**Remark 25.** It is instructive to examine the special case of a compact zero dimensional manifold, i.e. a finite set. In this case the dimension of \(E^0(M)\) is the cardinality of \(M\) and taking dimensions in the exact sequence of paragraph 21 gives the inclusion exclusion principle in combinatorics:

\[
\text{card}(U \cup V) = \text{card}(U) + \text{card}(V) - \text{card}(U \cap V)
\]

for finite sets \(U\) and \(V\).

**Theorem 26.** The maps \(H^k(M) \to H^k_s(M)\) and \(H^k_c(M) \to H^k_{cs}(M)\) induced by the chain map \(E^*(M) \to C^k(M)\) defined in paragraph 10 is an isomorphism.

**Proof.** (Sketch) We prove the theorem under the hypothesis that the manifold \(M\) is of finite type as explained below. This covers the compact case. With more care the argument can be made to work for any manifold. Form the diagram

\[
\cdots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to \cdots \\
\downarrow \downarrow \downarrow \\
\cdots \to H^k_s(U \cup V) \to H^k_s(U) \oplus H^k_s(V) \to H^k_s(U \cap V) \to \cdots
\]

where the rows are the two Mayer Vietoris sequences and the columns are the maps in cohomology induced from the chain map of paragraph 10. It is not hard to show (exercise) that this diagram commutes. From the five lemma we conclude that if the theorem is true for \(U, V\) and \(U \cap V\) then it is true for \(U \cup V\).

A manifold \(M\) is said to be of finite type iff there is a finite open cover \(\{U_i\}_i\) of \(M\) such that each nonempty finite intersection

\[
U_{i_1 \ldots i_k} := U_{i_1} \cap \cdots \cap U_{i_k}
\]

is diffeomorphic to \(\mathbb{R}^m\). We call these sets the cells of the cover. We prove the theorem for finite unions of cells by induction on the rank of the union: the rank of the union is defined as the smallest number \(r\) such that the union is a union of \(r\) cells. If \(U\) has rank \(r\) and \(V\) is a cell then \(U \cup V\) has rank \(\leq r + 1\). Hence by the above Mayer Vietoris argument it is enough to prove the theorem for cells, i.e. for \(\mathbb{R}^m\). But \(\mathbb{R}^m\) is contractible so \(H^k(\mathbb{R}^m) = H^k_s(\mathbb{R}^m) = 0\) for \(k > 0\) and \(H^0(\mathbb{R}^m) = H^0_{cs}(\mathbb{R}^m) = \mathbb{R}\). The map \(H^0(\mathbb{R}^m) \to H^0_{cs}(\mathbb{R}^m)\) is easily seen to be an isomorphism. The same argument (using
Mayer Vietoris sequences for compactly supported de Rham cohomology and compactly supported singular cohomology) works in the compactly supported case.

\[ \text{Theorem 27 (Poincaré Duality). Let } M \text{ be an orientable manifold of dimension } m \text{ without boundary. Then the map} \]

\[ E^*(M) \to E^{m-*}_c \omega \mapsto \left( \beta \mapsto \int_M \omega \wedge \beta \right) \]

is a chain map and induces isomorphisms \( H^k(M) \to H^{m-k}_c(M) \).

Proof. Mimic the proof of Theorem 26. Use the diagram

\[
\cdots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to \cdots \\
\downarrow \quad \downarrow \\
\cdots \to H^{m-k}_c(U \cup V) \to H^{m-k}_c(U) \oplus H^{m-k}_c(V) \to H^{m-k}_c(U \cap V) \to \cdots 
\]

\[ \text{Corollary 28. Let } M \text{ be a connected oriented manifold of dimension } m. \text{ Then } H^m_c(M) = \mathbb{R}. \text{ More precisely, a form } \omega \in E^m_c(M) \text{ is exact if and only if } \int_M \omega = 0. \]

Proof. \( H^m(M) = H^m_c(M) \) and \( H^0(M) = \mathbb{R} \). The second part follows since the map

\[ E^m_c(M) \to \mathbb{R} : \omega \mapsto \int_M \omega \]

induces a nonzero map \( H^m_c(M) \to \mathbb{R} \) as \( \int_M d\theta = 0 \) for \( \theta \in E^{m-1}_c(M) \).

\[ \text{Corollary 29. Let } M \text{ be a connected oriented manifold of dimension } m \text{ without boundary. If } M \text{ is noncompact then } H^m(M) = 0. \]

Proof. If \( M \) is noncompact, then \( H_0(M) = 0 \) because a nonzero constant function cannot have compact support.

\[ \text{Corollary 30. Let } M \text{ be a compact connected oriented manifold of dimension } m \text{ with nonempty boundary. Then } H^m(M) = 0. \]
Proof. It is not hard to prove that a manifold with boundary has a **collar neighborhood** This means that $M$ is (diffeomorphic to) a subset of a manifold $W$ without boundary and there is a diffeomorphism $\phi : (\partial M) \times J \rightarrow N$ where $N$ is an open subset of $W$ and $J = [-1, 1]$ such that $\phi(p, 0) = p$ for $p \in \partial M$, $\phi(p, t) \in M \setminus \partial M$ for $t < 0$ and $\phi \partial M \times V_+ = W \setminus M$ with $J_+ = ]0, 1[$. It follows easily that $W$ and $M$ are homotopy equivalent. Hence $H^m(M) = H^m(W) = 0$.

**Corollary 31.** Let $M$ be a connected nonorientable manifold of dimension $m$ with or without boundary and compact or not. Then $H_m^c(M) = H_m^c(W) = 0$.

**Proof.** It is not hard to construct a **orientable double cover** $\pi : W \rightarrow M$, i.e. $W$ is a connected orientable manifold, $\pi : W \rightarrow M$ is a local diffeomorphism, and $\pi^{-1}(p)$ consists of exactly two points for each $p \in M$. The involution $f : W \rightarrow W$ which interchanges the two points of the fiber is orientation reversing and satisfies $\pi \circ f = \pi$. (The archetypal example is $M = P^2, W = S^2$, and $f$ = the antipodal map.) For $\omega \in E^m(M)$ the form $\pi^*\omega \in E^m(W)$ is exact, i.e. $\pi^*\omega = d\theta$ for some $\theta \in E^{m-1}(W)$. If $M$ is noncompact or has a nonempty boundary the same is true of $W$ so this follows from $H^m(M) = 0$. If $M$ is compact and $\partial M = 0$ we note that

$$2 \int_W \pi^*\omega = \int_W \pi^*\omega + \int_W (\pi \circ f)^*\omega = \int_W \pi^*\omega - \int_W \pi^*\omega = 0$$

so $\pi^*\omega$ is exact by Corollary 28. The same reasoning shows that we can find $\theta$ of compact support if $\omega$ has compact support. Since $f^*\pi^*\omega = \pi^*\omega$ we may replace $\theta$ by $(\theta + f^*\theta)/2$ and assume w.l.o.g. that $f^*\theta = \theta$. But this means that $\theta = \pi^*\beta$ for some form $\beta \in E^{m-1}(M)$. Now $\pi^*(\omega - d\beta) = 0$ so $\omega = d\beta$ since $\pi$ is a local diffeomorphism. $\square$

32. We place the proof of Theorem 26 in a more general setting. A **finite simplicial complex** consists of a finite set called the set of vertices and a collection $\mathcal{N}$ of nonempty sets $\sigma$ of vertices called **simplices** such that (1) every singleton $\{i\}$ is a simplex and (2) every nonempty subset of a simplex is a simplex. A $k$-**simplex** is simplex with exactly $k + 1$ elements. Let $\mathcal{N}^k$ denote the set of $k$-simplices of $\mathcal{N}$. We do not distinguish the 0-simplex $\{i\}$ and the vertex $i$. We will take the set of vertices to be an initial segment $\{0, 1, \ldots, n\}$ of the non negative integers. The **geometric realization** of $\mathcal{N}$ is the set

$$|\mathcal{N}| = \{t \in \Delta^n : \text{supp}(t) \in \mathcal{N}\}$$
where supp$(t) = \{i : t_i \neq 0\}$ for $t = (t_0, t_1, \ldots, t_n) \in \Delta^n$.

**33.** An open cover $\mathcal{U} = \{U_i\}_{i=0}^n$ of a topological space $M$ determines a simplicial complex $\mathcal{N}$ called the **nerve** of the cover via the formula

$$\mathcal{N} = \{\sigma : U_\sigma \neq 0\}, \quad U_\sigma := \bigcap_{i \in \sigma} U_i.$$

Let $C_k(\mathcal{U})$ denote the vector space of formal sums

$$c = \sum_{\sigma \in \mathcal{N}^k} c_\sigma \sigma$$

or equivalently as the space of maps $\mathcal{N}^k \to \mathbb{R} : \sigma \mapsto c_\sigma$. Define a chain complex $\partial : \mathbb{C}_* \to \mathbb{C}_{*-1}$ by

$$\partial \sigma = \sum_{r=0}^{k} (-1)^r \sigma/r$$

where $\sigma/r$ denotes the $r$th face of $\sigma$, i.e.

$$\sigma/r = \{i_0, \ldots, i_{r-1}, i_{r+1}, \ldots, i_k\}$$

for $\sigma = \{i_0 < i_1 < \cdots < i_k\} \in \mathcal{N}^k$. The dual cochain complex $\delta : \mathbb{C}^* \to \mathbb{C}^{*+1}$ is defined by

$$\delta c = \sum_{r=0}^{k+1} (-1)^r c_{\tau/r}.$$

Here we view $c \in C^k(\mathcal{U})$ as a map $\mathcal{N}^k \to \mathbb{R} : \sigma \mapsto c_\sigma$. We call the homology and cohomology

$$H_k(\mathcal{U}) := \text{ker}(\partial)/\text{im}(\partial), \quad H^k(\mathcal{U}) := \text{ker}(\delta)/\text{im}(\delta)$$

**combinatorial homology** and **combinatorial cohomology** of the cover $\mathcal{U}$. Note that the boundary operator $\partial$ and its dual $\delta$ depend on the ordering of the vertices but the homology and cohomology groups do not.

**Remark 34.** A partition of unity $\{\kappa_i\}_{i=0}^n$ subordinate to cover $\mathcal{U}$ determines a map $\kappa : M \to |\mathcal{N}|$: the $i$th coordinate of the map is $\kappa_i$. The map $\kappa$ sends the set $U_i$ to the **open star**

$$|\mathcal{N}|_i := \{t \in |\mathcal{N}| : t_i > 0\}$$
of the $i$th vertex. These stars form a cover of $|\mathcal{N}|$ such each nonempty intersection

$$|\mathcal{N}|_{\sigma} := \bigcap_{i \in \sigma} |\mathcal{N}|_i$$

is contractible. We call the original cover $\mathcal{U}$ a **good cover** iff each cell $U_{\sigma}$ is contractible; it then follows that the map $\kappa : M \to |\mathcal{N}|$ is a homotopy equivalence. For any cover the map $\kappa$ induces isomorphisms from the combinatorial homology of the cover to the (continuous) singular homology cohomology of the geometric realization $|\mathcal{N}|$.

**Remark 35.** Refinements of covers determine chain maps between the corresponding combinatorial chain and cochain complexes. The limit homology is called the Čech (co)homology. The combinatorial homology and cohomology of a good cover is already equal to the Čech homology and cohomology. We will not develop this theory here.

**36.** Now we return to the situation of the proof of Theorem 26. The cells $U_{\sigma}$ are diffeomorphic to $\mathbb{R}^m$ and so are smoothly contractible. In particular, $\mathcal{U}$ is a good cover. We define the **de Rham Combinatorial double complex** as in Figure 1. In this diagram

$$\mathcal{E}^{p,q}(\mathcal{U}) := \bigoplus_{\sigma \in \mathcal{N}^q} \mathcal{E}^{p}(U_{\sigma})$$

and the maps $C^q(\mathcal{U}) \to \mathcal{E}^{0,q}(\mathcal{U})$ in the first column identify the real number $c_{\sigma} \in \mathbb{R}$ with the corresponding constant function in $\mathcal{E}^{0}(U_{\sigma})$.

**Theorem 37.** For a good cover $\mathcal{U}$ of a smooth manifold $M$ the zigzag relation of the de Rham Combinatorial double complex described in the proof below induces an isomorphism between the de Rham cohomology $H^k(M)$ and the combinatorial cohomology $H^k(\mathcal{U})$.

**Proof.** In the diagram of Figure 1 each rectangle commutes up to a sign, the rows are exact except for the bottom row and the columns are exact except for the leftmost one. The reason that the rows are exact is the Poincaré Lemma (Corollary 15). To prove that the columns are exact we construct a collection of chain homotopies

$$K : \mathcal{E}^{*,*}(\mathcal{U}) \to \mathcal{E}^{*,*+1}(\mathcal{U})$$
via the formula

\[ (K\omega)_\rho = \sum_{\sigma/r=\rho} (-1)^r \kappa_{r\sigma} \omega_{\sigma} \]

\( \rho \in \mathcal{N}^{q-1} \) and \( \omega \in \mathcal{E}^{p,q} \). (Recall that \( \sigma/r \) is the \( r \)th face of \( \sigma \).) In case \( q = 0 \) this formula is to be interpreted as

\[ K\omega = \sum_{i=0}^{n} \kappa_i \omega_i \in \mathcal{E}^p(M) \]

for \( \omega = \{\omega_i\}_{i=0}^{n} \in \mathcal{E}^{p,q}(\mathcal{U}) \). It is not hard to show that \( K \) is (up to a sign) chain homotopic to the identity, i.e.

\[ \omega = \pm K\delta\omega \pm \delta K\omega. \]

We say that \( c \in C^p(\mathcal{U}) \) and \( \omega \in \mathcal{E}^p(M) \) are **zig zag related** if there are elements \( \theta_{k,p-k} \in \mathcal{E}^{k,p-k}(\mathcal{U}) \) such that

\[ \delta\theta_{k,p-k} = d\theta_{k,p-k} \]

and

\[ c = \theta_{0,p}, \quad (\theta_{p,0})_\sigma = \omega|U_\sigma. \]
It is not hard to show that the set of all pairs \([c], [\omega]\) of cohomology classes such that \(c\) and \(\omega\) are zig zag related is the graph of an isomorphism between the combinatorial cohomology \(H^p(U)\) and the de Rham cohomology \(H^p(M)\). \(\Box\)