Theorem 1. Let \( f : W \to M \) be an injective immersion. Then \( f \) is an embedding if and only if \( f(W) \) is a submanifold of \( M \).

Proof. Assume that \( f \) is an embedding. Choose \( q_0 = f(p_0) \in f(W) \). We must find a submanifold chart \((U, \phi)\) at \( q_0 \) for \( f(W) \), i.e. \( \phi(U) = X \times Y \) where \( X \subset \mathbb{R}^r \) and \( Y \subset \mathbb{R}^k \) are open neighborhoods of the origin and \( \phi(U \cap f(W)) = X \times \{0\} \). By the implicit function theorem there is a neighborhood \( W_0 \) of \( p_0 \) in \( W \) such that \( f|W_0 \) is defines a diffeomorphism between \( W_0 \) and a submanifold \( f(W_0) \) of \( M \). Since \( f \) is an embedding, \( f(W_0) \) is a relatively open subset of \( f(W) \) in \( M \). Hence there is an open set \( M_0 \subset M \) with \( f(W_0) = M_0 \cap f(W) \). Then a submanifold chart \((U, \phi)\) for \( f(W_0) \) such that \( U \subset M_0 \) is a submanifold chart for \( f(W) \).

Conversely, assume that \( f(W) \) is a submanifold of \( M \). Then \( f \) (viewed as a map from \( W \) to \( f(W) \)) is a bijection and (by the inverse function theorem) a local diffeomorphism, in particular, a local homeomorphism. It follows that \( f \) is a homeomorphism onto its image, i.e. that \( f \) is an embedding. \(\Box\)

2. Let \( M \) be a manifold of dimension \( m \) and \( E \subset TM \) be a subbundle of its tangent bundle of rank \( r \) and corank \( k \), i.e. \( r + k = m \). An integral manifold for \( E \) is an injective immersion \( f : W \to M \) such that

\[(T_w f) T_w W = E_{f(w)}\]

for \( w \in W \). A submanifold \( W \subset M \) such that \( T_w W = E_w \) for \( w \in W \) is called an integral submanifold. Theorem 1 implies that an integral manifold \( f \) is an embedding if and only if its image \( f(W) \) is an integral submanifold. A foliation box for \( E \) is a diffeomorphism \( \phi : U \to X \times Y \) where \( U \) is an open
subset of $M$, $X$ and $Y$ are manifolds (of dimensions $r$ and $k$ respectively), and

$$(T_p \phi)E_p = T_x X \times \{0\} \subset T_x X \times T_y Y = T_{(x,y)}(X \times Y)$$

for $p \in U$, $\phi(p) = (x, y) \in X \times Y$. It follows that $\dim(X) = r$, $\dim(Y) = k$, and each slice $S_y = \phi^{-1}(X \times \{y\})$ for $y \in Y$ is an integral submanifold for $E$. It is a consequence of the Frobenius Complete Integrability Theorem that there is a foliation box at each point if and only if there is an integral manifold through each point. When these equivalent conditions hold we say that $E$ is an **integrable subbundle**.

**Lemma 3.** Let $f_1 : W_1 \to M$ and $f_2 : W_2 \to M$ be integral manifolds for integrable subbundle $E$. Then $f_1^{-1}(f_1(W_1) \cap f_2(W_2))$ is an open submanifold of $W_i$ and the overlap map

$$f_2 \circ f_1^{-1} : f_1^{-1}(f_1(W_1) \cap f_2(W_2)) \to f_2^{-1}(f_1(W_1) \cap f_2(W_2))$$

is a diffeomorphism.

**Remark 4.** The figure eight shows that Lemma 3 fails for injective immersions $f_1$ and $f_2$ even if we assume that $f_1(W_1) = f_2(W_2)$. A figure eight consisting of two simple closed curves tangent to infinite order cannot be an integral curve to a subbundle else there would be two orbits to a nowhere zero smooth section of the subbundle through the same point. However there is an example of a vectorfield with a zero and two homoclinic orbits whose closures are smooth circles tangent to each at the zero. (A homoclinic orbit is one which begins and ends at the same zero of the generating vector field.)

**Corollary 5.** Let $E$ be an integrable subbundle. Then through every point $p$ of $M$ there is a **maximal connected integral manifold**, i.e. an integral manifold $f : W \to M$ where $W$ is connected and $p \in f(W)$ and such that for every other integral manifold $f_1 : W_1 \to M$ with these properties $f_1(W_1) \subset f(W)$.

**Remark 6.** It is an immediate consequence of Lemma 3 that two maximal integral manifolds $f_i : W_i \to M$ through the same point are equivalent in the sense that $f_2 \circ f_1^{-1}$ is a diffeomorphism from $W_1$ onto $W_2$.

**Definition 7.** The image $f(W)$ of a maximal integral manifold is called a **leaf** and a decomposition of $M$ as a disjoint union of leaves of some integrable subbundle $E$ of $TM$ is called a **foliation** of $M$. 

Remark 8. The orbits of a flow without rest points are the leaves of a foliation; the corresponding subbundle is the trivial rank one subbundle which has the generating vector field as a nowhere zero section.

Example 9. The flow on the torus $T^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ defined by

$$
\phi^t(x, y) = (x + t, y + ta), \quad (x, y) \in T^2, \quad t \in \mathbb{R}
$$

has the property that every orbit is dense if $a$ is irrational.

Example 10. By elementary calculus the flow on the torus $T^2$ generated by the differential equations

$$
\dot{x} = 1, \quad \dot{y} = \cos(2\pi y)
$$

is

$$
\phi^t(x, y) = (x + t, f^{-1}(f(y)e^{2\pi t}))
$$

where $f(y) = \sec(2\pi y) + \tan(2\pi y)$ for $y \in \mathbb{R}/\mathbb{Z}$. The flow has two closed orbits at $y = \pm 1/4$. Any other orbit spirals towards the circle $y = \pm 1/4$ as $t \to \pm \infty$. In particular the orbit through $(x, y)$ is a closed subset of the torus if and only $(x, y)$ lies on one of the two circles $y = \pm 1/4$. Every orbit of this flow is a submanifold, i.e. the image of an embedding. Note that at a point $p_0$ of one of the two closed leaves (orbits) a sufficiently small foliation box $\phi : U \to X \times Y$ intersects the leaf through $p_0$ exactly in exactly one slice $\phi^{-1}(X \times \{y_0\})$ but the nearby leaves intersect $U$ in countably many slices.

Lemma 11. Let $f : W \to M$ be a leaf of a foliation of $M$ and assume that the image $f(W)$ is a closed subset of $M$. Then this leaf is a submanifold of $M$ and hence (by Lemma 1) $f$ is an embedding.

Proof. We show that at every point $p_0 \in f(W)$ there is a foliation box $\phi : U \to X \times Y$ such that $f(W) \cap U = \phi^{-1}(S \times \{y_0\})$ where $\phi(p_0) = (x_0, y_0)$, i.e. such that the leaf passes through the box exactly once. Define the set

$$
K = \{y \in Y : \phi^{-1}(X \times \{y\}) \subset f(W)\}.
$$

If we cannot shrink $U$ so that $K$ is a single point, then $y_0$ must be an accumulation point of $K$, i.e. there is a sequence of distinct points of $K$ converging to $y_0$ By a monodromy argument (see below) it follows that every point of $K$ is an accumulation point of $K$, i.e. $K$ is a perfect set. But this (and the
fact that $K$ is closed in $U$) implies (see below) that $K$ is uncountable, which contradicts the fact that $W$ is second countable and can there for contain at most countably many disjoint open sets.

Here is the aforementioned monodromy argument. Suppose that the leaf $f(W)$ through $p_0 = \phi^{-1}(x_0, y_0)$ accumulates on itself at $p_0$; we will show that it accumulates on itself at each of its points. More specifically, assume that $p_n \in f(W) \cap U$, and that the points $p_n$ lie in distinct components of $f(W) \cap U$ and satisfy $\lim_{n \to \infty} p_n = p_0$. Choose $\tilde{p}_0 \in f(W)$ and a foliation box $\tilde{\phi} : \tilde{U} \to \tilde{X} \times \tilde{Y}$ at $\tilde{p}_0$; we will construct points $\tilde{p}_n \in f(W) \cap \tilde{U}$ which lie in distinct components of $f(W) \cap \tilde{U}$ and satisfy $\lim_{n \to \infty} \tilde{p}_n = \tilde{p}_0$. (The assertion of the previous paragraph is the special case $\tilde{\phi} = \phi$.) Let $\gamma : [0, 1] \to W$ be a path from $f^{-1}(p_0)$ to $f^{-1}(\tilde{p}_0)$ and cover the image $f \circ \gamma$ of this path by finitely many foliation boxes. Hence, by induction, we may reduce the assertion to two special cases: (1) $\tilde{\phi} = \phi$ and $p_0$ and $\tilde{p}_0$ lie in the same component of $f(W) \cap U$, and (2) $\tilde{p}_0 = p_0$. In case (1) define $\tilde{p}_n$ by $\tilde{\phi}(\tilde{p}_n) = (x_n - x_0 + \tilde{x}_0, y_n)$ where $\phi(p_n) = (x_n, y_n)$, $\phi(p_0) = (x_0, y_0)$, and $\phi(\tilde{p}_0) = (\tilde{x}_0, y_0)$. In case (2) the problem is that points in the same component of $f(W) \cap U$ can be in different components of $f(W) \cap U$ because $U \cap \tilde{U}$ need not be connected. Choose a neighborhood $V$ of $p_0$ so that $V \subset U \cap \tilde{U}$ of form $\phi^{-1}(X_0, Y_0)$ where $X_0$ and $Y_0$ are connected neighborhoods of $x_0$ and $y_0$ in $X$ and $Y$ respectively. Then each component of $f(W) \cap U$ intersects $V$ in a connected set; if the component of $f(W) \cap U$ is $\phi^{-1}(X \times \{y\}$ the intersection is $V \cap f(W) = \phi^{-1}(X_0 \times \{y\})$ (or empty if $y \notin Y_0$. Each component of $f(W) \cap V$ is a connected subset of $f(W) \cap \tilde{U}$ and so lies in a component of $f(W) \cap \tilde{U}$. For $n$ sufficiently large we have that $p_n \in V$ so distinct points $p_n$ lie in distinct components of $f(W) \cap U$ and in distinct components of $f(W) \cap \tilde{U}$ as required.

Here is the proof that a perfect set $K$ is uncountable. Choose distinct points $y_0, y_1 \in K$. Choose disjoint neighborhoods $K_0$ and $K_1$ of $y_0$ and $y_1$ respectively. Since $y_0$ and $y_1$ are both accumulations points of $K$ there are distinct points $y_{00}, y_{01} \in K_0$ and $y_{10}, y_{11} \in K_1$; choose disjoint neighborhoods $K_{ij}$ of $x_{ij}$ for $i, j = 0, 1$. Repeat this process infinitely often. Then for every infinite sequence $(i_1, i_2, \ldots)$ of zeros an ones we get a decreasing sequence $K_{i_1} \supset K_{i_1 i_2} \supset \cdots$ of open sets. By shrinking the open sets we can assume that each has compact closure and that the closures of any two sets $K_{i_1 \ldots i_m}$ and $K_{j_1 \ldots j_n}$ are disjoint unless one of the sequences $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_n)$ is an initial segment of the other. By compactness there is a point in the infinite intersection $\bigcap_n K_{i_1 \ldots j_n}$. These points are distinct and (as $K$ is closed in $U$) they lie in $K$ as claimed. \qed
**Definition 12.** A section of a foliation of $M$ is a submanifold $S$ of form $S = \varphi^{-1}(\{x_0\} \times Y)$ where $\varphi: U \to X \times Y$ is a foliation box through which each leaf of the foliation passes at most once and $x_0 \in X$.

**Example 13.** Consider the flow on $S^3 = \{(x, y) \in \mathbb{C}^2 : x\bar{x} + y\bar{y} = 1\}$ defined by
$$
\varphi^t(x, y) = (e^{2it}x, e^{it}y).
$$
Each orbit is an embedded circle and is compact and hence closed. The leaf $S^1 \times 0$ passes once through a small foliation box about the point $(1, 0)$ but every nearby leaf passes through twice. There is no section through the point $(1, 0)$ but there is a section through each point $(x, y)$ with $y \neq 0$.

**Lemma 14.** Let $S_1$ and $S_2$ be sections for a foliation of $M$, let $R$ be the set of pairs $(p_1, p_2) \in S_1 \times S_2$ such that $p_1$ and $p_2$ lie in the same leaf, and let $Q_1 \subset S_1$ and $Q_2 \subset S_2$ be the projections of $R$ on $S_1$ and $S_2$ respectively. Then $Q_1$ and $Q_2$ are open in $S_1$ and $S_2$ respectively and $R$ is the graph of a diffeomorphism from $Q_1$ to $Q_2$.

**Proof.** That $R$ is the graph of a bijection between $Q_1$ and $Q_2$ is an immediate consequence of the definition of section. The rest follows from an argument like the monodromy argument used in the proof of Lemma 11.

**Example 15.** Lemma 14 says that if there is a section through every point of $M$ the space of leaves of the foliation forms a manifold with the sections as charts. However, this manifold need not be Hausdorff. For example, let $M = \{(x, y) \in \mathbb{R}^2 : x < |y|\}$, and let $\pi: M \to \mathbb{R}$ by $\pi(x, y) = x$. Then $\pi$ is a surjective submersion, but $\pi$ is not locally trivial since $\pi^{-1}(x)$ is connected for $x < 0$ and has of two components for $x \geq 0$. The space of leaves is the (non Hausdorff) manifold obtained from the disjoint union of two copies of $\mathbb{R}$ by identifying each negative number in the first copy with its mate in the second.

**Theorem 16.** Let $H$ be a closed subgroup of a Lie group $G$. Then $H$ is a submanifold of $G$.

**Proof.** Let $\mathfrak{g} = T_eG$ be the Lie algebra of $G$ and $\mathfrak{h}$ be the set of all $A \in \mathfrak{g}$ such that $\exp(tA) \in H$ for all $t \in \mathbb{R}$. In [1] page 110 it is shown that $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and for a sufficiently small neighborhood $U$ of $e$ in $G$ we have $\exp^{-1}(U \cap H) = \exp^{-1}(U) \cap \mathfrak{h}$. Let $\mathfrak{h}'$ be a vector space complement to $\mathfrak{h}$ in
\( \mathfrak{g} \) and shrink \( U \) so that \( \exp^{-1}(U) = X \times Y \) is (after the natural identification of \( \mathfrak{g} \) and \( \mathfrak{h} \times \mathfrak{h}' \)) the product of \( X = \exp^{-1}(U) \cap \mathfrak{h} \mathfrak{y} = \exp^{-1}(U) \cap \mathfrak{h}' \) and let \( \phi = \exp^{-1}|U \). Then \( \phi \) is a submanifold chart for \( H \) at the identity \( e \in H \). A submanifold chart at any other point of \( H \) can be constructed by translation (either left or right).

17. The left cosets \( gH \in G/H \) of a closed Lie subgroup \( H \) of a Lie group \( G \) form the leaves of foliation of \( G \). As in Lemma 14 this means that the space \( G/H \) is a manifold. In [1] page 120 it is shown that \( G/H \) is Hausdorff and that the projection \( G \rightarrow G/H \) is locally trivial.

References