The Grassmann Manifold

1. For vector spaces V and W denote by $\mathbf{L}(V, W)$ the vector space of linear maps from V to W. Thus $\mathbf{L}(\mathbb{R}^k, \mathbb{R}^n)$ may be identified with the space $\mathbb{R}^{k\times n}$ of $k \times n$ matrices. An injective linear map $u : \mathbb{R}^k \to V$ is called a k-**frame** in V. The set

$$\mathrm{GF}_{k,n} = \{u \in \mathbf{L}(\mathbb{R}^k, \mathbb{R}^n) : \mathsf{rank}(u) = k\}$$

of k-frames in $\mathbb{R}^n$ is called the **Stiefel manifold**. Note that the special case $k = n$ is the **general linear group**:

$$\mathrm{GL}_k = \{a \in \mathbf{L}(\mathbb{R}^k, \mathbb{R}^k) : \det(a) \neq 0\}.$$

The set of all k-dimensional (vector) subspaces $\lambda \subset \mathbb{R}^n$ is called the **Grassmann manifold** of k-planes in $\mathbb{R}^n$ and denoted by $\mathrm{GR}_{k,n}$ or sometimes $\mathrm{GR}_{k,n}(\mathbb{R})$ or $\mathrm{GR}_k(\mathbb{R}^n)$. Let

$$\pi : \mathrm{GF}_{k,n} \to \mathrm{GR}_{k,n}, \qquad \pi(u) = u(\mathbb{R}^k)$$

denote the map which assigns to each k-frame u the subspace $u(\mathbb{R}^k)$ it spans. For $\lambda \in \mathrm{GR}_{k,n}$ the fiber (preimage) $\pi^{-1}(\lambda)$ consists of those k-frames which form a basis for the subspace λ, i.e. for any $u \in \pi^{-1}(\lambda)$ we have

$$\pi^{-1}(\lambda) = \{u \circ a : a \in \mathrm{GL}_k\}.$$

Hence we can (and will) view $\mathrm{GR}_{k,n}$ as the orbit space of the group action

$$\mathrm{GF}_{k,n} \times \mathrm{GL}_k \to \mathrm{GF}_{k,n} : (u, a) \mapsto u \circ a.$$

The exercises below will prove the following

Theorem 2. *The Stiefel manifold* $\mathrm{GF}_{k,n}$ *is an open subset of the set* $\mathbb{R}^{n\times k}$ *of all* $n \times k$ *matrices. There is a unique differentiable structure on the Grassmann manifold* $\mathrm{GR}_{k,n}$ *such that the map* π *is a submersion.*

3. Show that $\mathrm{GF}_{k,n}$ is an open subset of $\mathbb{R}^{n\times k}$. Hint: For any k-element subset

$$I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, 2, \ldots, n\},$$

define a linear projection $p_I : \mathbb{R}^n \to \mathbb{R}^k$ by

$$p_I(x_1, x_2, \ldots, x_n) = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}).$$

For $u \in \mathbf{L}(\mathbb{R}^m, \mathbb{R}^n)$ the matrix

$$u_I = p_I \circ u \in \mathbf{L}(\mathbb{R}^m, \mathbb{R}^k)$$

is obtained from u be deleting the columns not corresponding to elements of I. Let

$$V_I = \{u \in \mathbb{R}^{n\times k} : \det(u_I) \neq 0\}.$$

The $\binom{n}{k}$ sets V_I cover $\mathrm{GF}_{k,n}$.

4. For any subset $I \subset \{1, 2, \dots, n\}$ of indices let $I' = \{1, 2, \dots, n\} \setminus I$ denote its complement. Let $U_I = \pi(V_I)$. Show that

$$V_I = \pi^{-1}(U_I),$$

that the matrix

$$\phi_I(\lambda) = u_{I'} u_I^{-1} \in \mathbb{R}^{k \times (n-k)}$$

is independent of the choice of $u \in \pi^{-1}(\lambda)$ used to define it, and that the map $\phi_I : U_I \to \mathbb{R}^{k \times (n-k)}$ is a bijection. The inverse is defined by

$$\phi_I^{-1}(A) = \{x \in \mathbb{R}^n : x_{I'} = A x_I\}$$

where we have written $x_I = \pi_I(x)$ and $x_{I'} = \pi_{I'}(x)$.

5. Show that the $\binom{n}{k}$ charts (ϕ_I, U_I) form a C^∞ atlas. Hint: The condition $A \in \phi_I(U_I \cap U_J)$ holds if and only if a certain submatrix of A is invertible.

6. Show that the map $\pi : \mathrm{GF}_{k,n} \to \mathrm{GR}_{k,n}$ is locally trivial. More precisely, show that for each k-element subset I there is a unique smooth map $\sigma_I : U_I \to V_I$ such that $\pi \circ \sigma_I$ is the identity map of U_i and

$$\sigma_I(U_I) = \{u \in \mathrm{GF}_{k,n} : u_I = 1_k\}.$$

Here 1_k denotes the $k \times k$ identity matrix.) Then show that the map $\Phi_I : U_I \times \mathrm{GL}_k \to V_I$ defined by

$$\Phi_I(\lambda, a) = \sigma_I(\lambda) \circ a$$

is a diffeomorphism satisfying $\pi \circ \Phi_I(\lambda, a) = \lambda$.

7. Suppose E, M_1, M_2 are smooth manifolds and that $p_i : E \to M_i$ are surjective submersions. Suppose that $f : M_1 \to M_2$ is a bijection such that $f \circ p_1 = p_2$. Show that f is a diffeomorphism.

8. Suppose that λ_0 is a k dimensional subspace of $\mathbb{R}^n$ and let μ_0 be any (vector space) complement to λ_0 in $\mathbb{R}^n$ i.e.,

$$\mathbb{R}^n = \lambda_0 \oplus \mu_0.$$

For $A \in \mathbf{L}(\lambda_0, \mu_0)$ let

$$\mathsf{Graph}(A) = \{x + Ax : x \in \lambda_0\}.$$

and let $\pi_0 : \mathbb{R}^n \to \lambda_0$ be the projection along μ_0. Show that the following conditions on k dimensional subspace λ of $\mathbb{R}^n$ are equivalent:

(a) $\lambda = \mathsf{Graph}(A)$ for some $A \in \mathbf{L}(\lambda_0, \mu_0)$;

(b) λ is transverse to μ_0 (meaning that $\lambda \cap \mu_0 = \{0\}$);

(c) $\pi_0 | \lambda$ is an isomorphism.

9. Let $\mathbb{R}^n = \lambda_0 \oplus \mu_0$ be a splitting as in paragraph 8 and let U be the set of all $\lambda \in \mathrm{GR}_{k,n}$ satisfying the equivalent conditions (a), (b), and (c). Define $\phi : U \to \mathbf{L}(\lambda_0, \mu_0)$ by

$$\phi^{-1}(A) = \mathsf{Graph}(A).$$

Prove that U is an open dense subset of $\mathrm{GR}_{k,n}$ and that ϕ is a diffeomorphism.

10. The manifold $\mathrm{GR}_{k,n}$ is an example of a homogeneous space. Prove this by showing:

(1) For each $a \in \mathrm{GL}_n$ the induced map $\mathrm{GR}_{k,n} \to \mathrm{GR}_{k,n} : \lambda \mapsto a(\lambda)$ is a diffeomorphism.

(2) The map $\mathrm{GL}_n \times \mathrm{GR}_{k,n} \to \mathrm{GR}_{k,n} : (a, \lambda) \mapsto a(\lambda)$ is smooth.

(3) For each $\lambda_0 \in \mathrm{GR}_{k,n}$ the map $\mathrm{GL}_n \to \mathrm{GR}_{k,n} : a \to a(\lambda_0)$ is a C^∞ surjective map.

Remark 11. If we take $\lambda_0 = \mathbb{R}^k \times 0$ in the map $a \to a(\lambda_0)$ the fibers of this map (i.e., inverse images of points) are left cosets $a \cdot \mathrm{GL}_{n,k}$ where $\mathrm{GL}_{n,k}$ is the subgroup of GL_n consisting of all $a \in \mathrm{GL}_n$ such that $a(\mathbb{R}^k \times 0) = \mathbb{R}^k \times 0$. Thus we have a bijective correspondence between $\mathrm{GR}_{k,n}$ and the space of left cosets $\mathrm{GL}_n/\mathrm{GL}_{n,k}$:

$$\mathrm{GR}_{k,n} \simeq \mathrm{GL}_n/\mathrm{GL}_{n,k}.$$

(Warning: $\mathrm{GL}_{n,k}$ is not a normal subgroup of GL_n. Thus $\mathrm{GR}_{k,n}$ is not a group in any natural fashion.)

12. Prove that the map $\mathrm{GR}_{k,n} \to \mathrm{GR}_{n-k,n} : \lambda \to \lambda^\perp$ (where $\lambda^\perp$ denotes the orthogonal complement to λ with respect to the usual inner product on $\mathbb{R}^n$) is a C^∞ diffeomorphism.