This course will explain the fundamental concepts of topology for smooth manifolds. After developing the fundamental tool of transversality theory (we prove Sard’s theorem) we will prove the Brouwer Fixed Point Theorem, the Brouwer No Retraction Theorem, and the Whitney Embedding Theorem, and the “27 lines on a cubic surface” using this tool. Then we cover Morse Theory and use it to classify 2-manifolds and prove the Bott Periodicity Theorem and Lefschetz Hyperplane Theorem. If time permits we will explain Smale’s proof of the higher dimensional Poincaré Conjecture.

1 Manifolds

1.1. Let $M$ be a set. A chart on $M$ is a pair $(\alpha, U)$ where $U \subset M$, $\alpha$ is a bijection from $U$ to an open subset of $\mathbb{R}^m$; i.e. $\alpha(U) \subset \mathbb{R}^m$ is open and $\alpha : U \to \alpha(U)$ is a bijection. Two charts $(\alpha, U)$ and $(\beta, V)$ are $C^r$ compatible iff $\alpha(U \cap V)$ and $\beta(U \cap V)$ are both open and

$$\beta \circ \alpha^{-1} : \alpha(U \cap V) \to \beta(U \cap V)$$

is a $C^r$ diffeomorphism. A $C^r$ atlas on $M$ is a collection $\mathcal{A}$ of charts on $M$ any two of which are $C^r$ compatible and such that the sets $U$, as $(\alpha, U)$ ranges over $\mathcal{A}$, cover $M$ (i.e., for every $x \in M$ there is a chart $(\alpha, U) \in \mathcal{A}$ with $x \in U$). A maximal $C^r$ atlas is an atlas which contains every chart which is $C^r$ compatible with each of its members. A maximal $C^r$ atlas is also called a $C^r$ structure. If $\mathcal{A}$ is an $C^r$ atlas, then so is the collection of all charts $C^r$ compatible with each member of $\mathcal{A}$. In other words, every $C^r$ atlas extends uniquely to a maximal $C^r$ atlas.
Definition 1.2. A $C^r$ manifold is a pair consisting of a set $M$ and a maximal $C^r$ atlas $A$ on $M$.

Remark 1.3. A $C^r$ manifold is usually specified by giving its underlying set $M$ and some $C^r$ atlas on $M$. Generally, the notation for the atlas is suppressed and the manifold is denoted simply by $M$. The members of the atlas are called the charts $M$.

Remark 1.4. A $C^r$ atlas is a $C^s$ atlas for $s \leq r$; hence every $C^r$ manifold is a $C^s$ manifold.

Definition 1.5. The manifold topology of a $C^r$ manifold $M$ is the topology generated by the sets $U$ as $(\alpha, U)$ ranges over the charts of $M$.

Example 1.6. The manifold topology need not be Hausdorff.

Example 1.7. The manifold topology need not be second countable.

Remark 1.8. Henceforth, unless otherwise stated, all manifolds will be assumed to be Hausdorff and second countable.

Definition 1.9. Let $M$ and $N$ be $C^r$ manifolds and

$$f : M \rightarrow N$$

be a map. Given charts $(\alpha, U)$ on $M$ and $(\beta, V)$ on $N$ we define the local representative

$$f_{\beta\alpha} : \alpha(U \cap f^{-1}(V)) \rightarrow \beta(V)$$

of $f$ with respect to $(\alpha, U)$ and $(\beta, V)$ by

$$f_{\beta\alpha} = \beta \circ f \circ \alpha^{-1}|\alpha(U \cap f^{-1}(V)).$$

The map $f$ is $C^r$ iff for all $C^r$ charts $(\beta, V)$ on $N$ and $(\alpha, U)$ on $M$ the set $U \cap f^{-1}(V)$ is open in $M$ and the local representative $f_{\beta\alpha}$ is $C^r$. Note that a map is $C^r$ if for every $x \in M$ there exist charts $C^r$ charts $(\beta, V)$ and $(\alpha, U)$ with $x \in U \cap f^{-1}(V)$, $U \cap f^{-1}(V)$ open, and $f_{\beta\alpha}$ is $C^r$.

1.10. The $C^r$ manifolds and $C^r$ maps form a category, i.e. the identity map on a $C^r$ manifold is a $C^r$ map, and the composition of $C^r$ maps is $C^r$. The the isomorphisms of this category are called $C^r$ diffeomorphisms. Thus a map $f : M \rightarrow N$ between $C^r$-manifolds is a $C^r$ diffeomorphism iff $f$ is
bijective and both \( f \) and \( f^{-1} \) are \( C^r \). Two \( C^r \) manifolds \( M \) and \( N \) are \( C^r \) \textbf{diffeomorphic} iff there exists a \( C^r \) diffeomorphism from \( M \) onto \( N \). It follows that a \( C^r \) map is \( C^s \) for \( s \leq r \) and map between \( C^r \) manifolds is a \( C^0 \) map if and only if it is continuous. If \( r \geq 1 \) then a \( C^r \) map which is a \( C^1 \) diffeomorphism is a \( C^r \) diffeomorphism, but a \( C^\infty \) homeomorphism need not be a \( C^1 \) diffeomorphism, for example, \( \mathbb{R} \to \mathbb{R} : x \mapsto x^3 \).

1.11. Let \( M \) be a manifold. Then \( M \) has \textbf{dimension} \( m \) if \( \alpha(U) \subset \mathbb{R}^m \) for every chart \((\alpha, U)\) of \( M \). Some authors call a manifold of dimension \( m \) an \( m \)-\textbf{manifold}.

\textbf{Remark 1.12.} Suppose \( U \subset \mathbb{R}^m \) and \( V \subset \mathbb{R}^n \) are \( C^r \) diffeomorphic then \( m = n \). For \( r \geq 1 \) this is quite easy to see, for if \( \phi : U \to V \) is a diffeomorphism then \( D\phi(x) : \mathbb{R}^m \to \mathbb{R}^n \) is a vector space isomorphism for \( x \in U \). For \( r = 0 \) (i.e. if \( \phi \) a homeomorphism) this is Brouwer’s famous \textit{invariance of domain} theorem. It follows that a connected manifold has dimension \( m \) for some \( m \).

\textbf{Example 1.13.} Let \( M_1 \) be the \( C^r \) manifold having the real line \( \mathbb{R} \) as its underlying subset and having as a \( C^r \) atlas the set consisting of the single chart \((\alpha, U)\) where \( U = M_1 = \mathbb{R} \) and \( \alpha(x) = x \) for \( x \in U \) and let \( M_2 \) be the \( C^r \) manifold having the real line \( \mathbb{R} \) as its underlying subset and having as a \( C^r \) atlas the set consisting of the single chart \((\beta, V)\) where \( V = M_2 = \mathbb{R} \) and \( \beta(y) = y^3 \) for \( y \in V \). If \( r = 0 \), these manifolds are the same (i.e., the corresponding maximal atlas are set theoretically the same) while if \( 1 \leq r \leq \infty \) they are distinct. In any case, they are \( C^r \) diffeomorphic; a diffeomorphism given by \( f : M_2 \to M_1 : f(z) = z^3 \).

\textbf{Remark 1.14.} More generally, two \( C^r \) manifolds having the same underlying set are the same (i.e., the corresponding maximal \( C^r \) atlas are identical) if and only if the identity map (considered as a map from the manifold to the other) is a \( C^r \) diffeomorphism. Even if the identity map is not a \( C^r \) diffeomorphism the manifolds may still be diffeomorphic via another map. Milnor’s famous example shows that two \( C^\infty \) manifolds can be homeomorphic but not \( C^\infty \) diffeomorphic. On the other hand, for \( r \geq 1 \) \( C^r \) manifolds are \( C^r \) diffeomorphic if and only if they are \( C^1 \) diffeomorphic. (See [7] for example.)

1.15. \textbf{Exercise.} Let \( \mathbb{R} \) have its usual \( C^r \) structure. Show that any \( C^r \) manifold which is homeomorphic to \( \mathbb{R} \) is \( C^r \) diffeomorphic to \( \mathbb{R} \).
2 Simplicial complexes

2.1. A simplicial complex is a collection $K$ of nonempty finite sets such that

$$\sigma \in K \text{ and } \emptyset \neq \tau \subset \sigma \implies \tau \in K.$$  

The points $v$ such that $\{v\} \in K$ are called the set of vertices of the complex $K$ and the elements of $K$ are called simplices of the complex. A when $\tau, \sigma \in K$ and $\tau \subset \sigma$ we say that $\tau$ is a face of $\sigma$. A simplex of cardinality $m + 1$ is called an $m$-simplex and for each integer $n \geq 0$ the set of simplices of cardinality $\leq n$ is called the $n$-skeleton of $K$ and denoted by $K_n$. The $0$-skeleton and the set $V$ of vertices are essentially the same:

$$K_0 = \{ \{v\} : v \in V \}.$$

To keep life simple we restrict attention to finite dimensional simplicial complexes meaning that

- (a) $K = K_m$ for some integer $m$;
- (b) the set of vertices is finite or countable;
- (c) for each vertex lies in only finitely many simplices.

Any finite complex has these properties. When $m$ is the smallest integer satisfying (a) we say that $K$ is $m$-dimensional.

2.2. Let $K$ be a simplicial complex and $V$ be its set of vertices. The geometric realization $|K|$ of the simplicial complex $K$ is the set of all functions $x : V \to [0, 1]$ such that

$$\{v \in V : x(v) > 0\} \in K$$

and

$$\sum_{v \in V} x(v) = 1.$$

We equip $|K|$ with the topology it inherits as a subset of the product space $[0, 1]^V$. By Tychonoff’s Theorem the product space is compact. However, if $K$ is infinite, the subset $|K|$ is not closed. For example, the functions $\{\delta_v\}_{v \in V}$ defined by $\delta_v(v) = 1$ and $\delta_v(w) = 0$ for $w \neq v$ form a discrete subset of $|K|$.  

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Theorem 2.3. Let $K$ be an $m$-dimensional simplicial complex. Then there is a proper topological embedding of $|K|$ in $\mathbb{R}^{2m+1}$, i.e. a continuous map $f : |K| \to \mathbb{R}^{2m+1}$ such that the image $f(|K|)$ is a closed subset of $\mathbb{R}^{2m+1}$ and $f$ is a homeomorphism from $|K|$ onto $f(|K|)$.

Proof. Choose an injective map $f : V \to \mathbb{R}^{2m+1}$ so that $f(V)$ is discrete, say $|f(v) - f(w)| > 1$ for $v \neq w$. Then perturb $f$ so that $\text{Span}(f(\sigma)) \cap \text{Span}(f(\tau)) = \text{Span}(f(\sigma \cap \tau))$ for $\sigma, \tau \in K$. Finally extend $f$ to $|K|$ via the formula $f(x) = \sum_{v \in V} x(v)v$. $\square$

2.4. Let $K$ and $K'$ be simplicial complexes with vertex sets $V$ and $V'$ respectively. A map $f : V \to V'$ is called a simplicial map iff for every simplex $\sigma \in K$ the set $f(\sigma) = \{f(v) : v \in \sigma\}$ is a simplex of $K'$. The set of simplicial complexes and simplicial maps between form a category meaning that the identity map is simplicial and the composition of simplicial maps is simplicial. An the operations $K \mapsto |K|$ and $f \mapsto |f|$ are functorial, i.e. they respect composition and send identity maps to identity maps. A map $f$ is called a simplicial isomorphism iff it is an isomorphism in this category, i.e. $f$ is a bijection and both $f$ and $f^{-1}$ are simplicial. The geometrical realization of the simplicial map $f$ is the map $|f| : |K| \to |K'|$ defined by $|f|(x)(v') = \sum_{f(v) = v'} x(v)$ for $x \in |K|$ and $v' \in V'$. A subdivision of a simplicial complex $K$ is a simplicial complex $K'$ such that (a) $V' \subset |K|$, (b) for every $\sigma' \in K'$ there is a $\sigma \in K$ with $\sigma' \subset |\sigma|$, and (c) the map $\iota : |K'| \to |K|$ defined by $\iota(x') = \sum_{v' \in V'} x'(v')v'$ is a homeomorphism. (The map $\iota$ is not the geometrical realization of a simplicial map.) A triangulation of a topological space $M$ is a simplicial complex $K$ together with a homeomorphism $\phi : |K| \to M$. Two triangulations are called combinatorially equivalent iff they have isomorphic subdivisions. A long standing conjecture called the Hauptvermutung said that any two triangulations of the same topological space are combinatorially equivalent. Milnor gave a counterexample to this and Kirby and Siebenmann gave counterexamples which are manifolds. However, for $r \geq 1$, a $C^r$ manifold has a triangulation and a version of the Hauptvermutung is true. See [7].
Example 2.5. The space $\mathbb{R}^m$ is triangulable, i.e. that there is a simplicial complex $K$ such that $|K|$ is homeomorphic to $\mathbb{R}^m$. For this we take the set of vertices to be $V = \mathbb{Z}^m$ and we label the simplices by pairs $(\nu, \pi)$ where $\nu \in \mathbb{Z}^m$ and $\pi$ is a permutation of $\{1, 2, \ldots, m\}$ as follows. Let $\sigma_\pi$ be the set of all points $e = (e_1, e_2, \ldots, e_m) \in \{0, 1\}^m$ such that $e_{\pi(1)} \leq e_{\pi(2)}, \ldots, e_{\pi(m)}$; the convex hull of $\sigma_\pi$ is the $m$-simplex $|\sigma_\pi| = \{(y_1, y_2, \ldots, y_m) : 0 \leq y_{\pi(1)} \leq y_{\pi(2)}, \ldots, y_{\pi(m)} \leq 1\}$.

Let $K$ be the set of all nonempty subsets $\sigma$ of $\mathbb{Z}^m$ such that $\sigma \subset \nu + \sigma_\pi$ for some $\nu \in \mathbb{Z}^m$ and some permutation $\pi$. Exercise. Prove that the map $|K| \to \mathbb{R}^m : x \mapsto \sum_{v \in \mathbb{Z}^m} x(v)v$ is a homeomorphism.

3 Partitions of unity

Definition 3.1. A $C^r$ partition of unity on $M$ is a collection $\{g_i\}_i$ of $C^r$ real valued functions on $M$ such that

1. $g_i \geq 0$ for each $i$;
2. every $x \in M$ has a neighborhood $U$ such that $U \cap \text{support}(g_i) = \emptyset$ for all but finitely many of the $g_i$;
3. for each $x \in M$ $\sum_i g_i(x) = 1$.

A partition of unity $\{g_i\}_i$ on $M$ is subordinate to an open cover of $M$ iff for each $g_i$ there is an element $U$ of the cover such that $\text{support}(g_i) \subset U$.

Remark 3.2. A continuous partition of unity is essentially the same thing as a map to the geometric realization of a simplicial complex. If $(V, K)$ is a simplicial complex, then the functions $\{g_v\}_{v \in V}$ on $|K|$ defined by $g_v(x) = x(v)$ satisfies conditions (1-3); if $f : M \to |K|$ is continuous, the functions $\{g_v \circ f\}_{v \in V}$ form a partition of unity on $M$. Every partition of unity $\{g_i\}_{i \in I}$ arises this way as follows. Take $V = I$ and define $K$ to be the set of all $\sigma$ such that $\bigcup_{i \in \sigma} g_i^{-1}([0, 1]) \neq \emptyset$. Then define $f : M \to |K|$ by $f(z)(i) = g_i(z)$.
3.3. Assume that $M$ is a $C^r$ manifold. Since $M$ is second countable there is a countable basis for the open sets. Using this countable basis construct a new countable basis $\{W_n\}_n$ such that each closure $W_n$ is a rectangle whose closure is contained in some coordinate system. (Choose $p \in M$, then a chart $(W, \alpha)$ with $p \in W$, then an element $V$ of the countable basis with $p \in V \subset W$, then a rectangle $U$ with rational coordinates and $p \in U \subset \bar{U} \subset V$, and let $\{U_n\}_n$ enumerate these rectangles.) Then there is a collection $\{f_n\}_n \subset C^r(M)$ with $f_n : M \to [0, \infty)$ and

$$U_n = \{x \in M : f_n(x) > 0\}.$$  

To construct $f_n$ it is enough to prove that every open rectangle in $\mathbb{R}^m$ has form $f^{-1}(0, \infty)$ where $f : \mathbb{R}^m \to [0, 1]$ is $C^\infty$. By multiplying the coordinates it is enough to do this when $m = 1$. The function $g(x) = e^{-1/x}$ for $x > 0$ (with $g(x) = 0$ for $x \leq 0$ is positive on $(0, \infty)$, the function $h(x) = g(x)g(1-x)$ is positive on $(0, 1)$, and the function $h(a + (b-a)x)$ is positive on $(a, b)$.

**Lemma 3.4.** Any closed subset of a $C^r$ manifold is the zero set of a $C^r$ real-valued function.

*Proof.* Let $A \subset M$ be closed. Let $I$ be the set of all $n$ such that $f_n = 0$ on $A$ where $f_n$ is as in paragraph 3.3. Let $g = \sum_{n \in I} c_n f_n$ where the positive constants $c_n$ converge to zero so rapidly that the series converges $C^r$ uniformly on compact sets; use diagonalization to construct $c_n$. Then $A = g^{-1}(0)$. \hfill $\Box$

**Theorem 3.5.** Let $M$ a $C^r$ manifold. Then for every open cover there is a $C^r$ partition of unity subordinate to it.

*Proof.* Suppose an open cover $\mathcal{U}$ is given. For each $p$ in $M$ there is a set $U \in \mathcal{U}$ of the cover and a set $U_n$ from the basis of paragraph 3.3 such that $p \in U_n \subset \bar{U}_n \subset U$. For each such $U_n$ let

$$V_n = \left\{ x \in U_n : f_k(x) < \frac{1}{n} \text{ for } 1 \leq k < n \right\}.$$  

The sets $\{V_n\}_n$ form a cover since for each point of $M$ there is a least $n$ with $p \in U_n$, i.e. $f_n(p) > 0$ and $f_k(p) = 0$ for $k < n$.

We prove that the cover $\{V_n\}_n$ is locally finite, i.e. that every point has a neighborhood intersecting only finitely many closures $\bar{V}_n$. Choose $p \in M$. Since the sets $\{V_k\}_k$ cover $M$, there exists $k$ such that $f_k(p) > 0$. Let $V =$
\{ q \in M : f_k(p) < 2 f_k(q) \}. Then \( p \in V \). If \( q \in V \cap \bar{V}_n \) and \( n > k \), then
\[ f_k(p) < 2 f_k(q) \leq 2/n \; \text{hence} \; V \cap \bar{V}_n = \emptyset \; \text{for} \; n > \max\{k, 1/f_k(p)\}, \] i.e. \( V \) intersects only finitely many sets \( \bar{V}_n \). By Lemma 3.4
\[ V_m = \{ x \in M : h_n(x) > 0 \} \]
where \( h_m \in C^\infty(M), h_m : M \to [0, \infty) \). Let
\[ g_m(x) = \frac{h_m(x)}{\sum_k h_k(x)} . \]
Then \( \{ g_m \} \) is the required partition of unity.

**Corollary 3.6 (\( C^\infty \) Urysohn’s Lemma).** Let \( X \) and \( Y \) be disjoint closed subsets of a \( C^\infty \) manifold \( M \). Then there is a \( C^\infty \) function \( h : M \to \mathbb{R} \) such that \( 0 \leq h(x) \leq 1 \) for all \( x \in M \), \( h(x) = 0 \) for \( x \in X \) and \( h(x) = 1 \) for \( x \in Y \).

*Proof.* Let \( \{ g_1, g_2 \} \) be a partition of unity subordinate to the open cover \( \{ U_1, U_2 \} \) where \( U_1 = M \setminus Y \) and \( U_2 = M \setminus X \). \( \square \)

**Corollary 3.7.** Given a finite open cover \( \{ U_i \}_{i=1}^n \) there exist \( C^\infty \) functions \( \{ h_i \}_{i=1}^n \) on \( M \) with \( h_i \) supported in \( U_i \) and such that the closed sets \( \{ h_i^{-1}(1) \}_{i=1}^n \) cover \( M \).

*Proof.* Choose a partition of unity \( \{ g_i \}_{i=1}^n \) subordinate to the cover. Let \( A_i = g_i^{-1}([1/n, \infty)) \) and choose \( h_i \) with \( h_i = 0 \) on \( M \setminus U_i \) and \( h_i = 1 \) on \( A_i \). \( \square \)

**Corollary 3.8.** Any \( C^\infty \) manifold \( M \) admits a proper (i.e. the preimage of a compact set is compact) \( C^\infty \) function \( g : M \to \mathbb{R} \).

*Proof.* Choose countable collections \( \{ U_n \}_{n=1}^\infty \) and \( \{ K_n \}_{n=1}^\infty \) such that the sets \( U_n \) are open, the collection \( \{ U_n \}_{n=1}^\infty \) is locally finite, the sets \( K_n \) are compact, \( K_n \subset U_n \), and \( M = \bigcup_{n=1}^\infty K_n \). For example, we can construct these collections from a countable partition of unity with compact supports by taking \( U_n \) to be the set where the \( n \)th function is nonzero, then taking a second partition of unity subordinate the cover \( \{ U_n \}_{n=1}^\infty \), and then taking \( K_n \) to be the support of the \( n \)th function in that second partition of unity. Then choose \( C^\infty \) functions \( h_n : M \to [0, 1] \) with \( h_n = 1 \) on \( K_n \) and \( h_n = 0 \) in \( M \setminus U_n \). Finally take \( g = \sum_{n=1}^\infty n h_n \). Then \( g^{-1}([0, r]) \) is subset of the compact set \( \bigcup_{n=1}^r K_n \) so the function \( g \) is proper. \( \square \)
Theorem 3.9 (Easy Whitney). Every smooth manifold can be properly embedded as a (necessarily closed) submanifold of \( \mathbb{R}^n \).

Proof. By Corollary 4.15 below \( M \) can be covered by finitely many charts \( \{(\alpha_i, U_i)\}_{i=1}^n \). Choose a partition of unity \( \{g_i\}_{i=1}^n \) subordinate to the cover and let \( A_i = g_i^{-1}((n+1)^{-1}, 1] \) and \( B_i = g_i^{-1}((n+2)^{-1}, 1] \). Then \( M = \bigcup_{i=1}^n A_i \) and \( A_i \subset B_i \subset \bar{B}_i \subset U_i \). By Urysohn’s Lemma (Corollary 3.6) let \( h_i \) be a \( C^r \) function such that \( h_i = 1 \) on \( \bar{B}_i \) and \( h_i = 0 \) on \( M \setminus U_i \). By Corollary 3.8 let \( g : M \to \mathbb{R} \) be a positive proper \( C^r \) function. Define \( f : M \to \mathbb{R}^{mn+n+1} \) by

\[
f = (h_1 \alpha_1, \ldots, h_n \alpha_n, h_1, \ldots, h_n, g).
\]

If \( f(p) = f(q) \) then \( h_i(p) = h_i(q) = 1 \) for some \( i \) so \( \alpha_i(p) = \alpha_i(q) \) so \( p = q \) so \( f \) is injective. Since \( dh_i = 0 \) on the open set \( B_i \) we have that \( d(h_i \alpha_i)(p) = d\alpha_i(p) \) for \( p \in A_i \) so \( f \) is an immersion. Any compact set \( K \) of \( \mathbb{R}^{mn+n+1} \) is contained in some set \( \mathbb{R}^{mn+m} \times J \) where \( J \in \mathbb{R} \) is a compact interval. Then \( f^{-1}(K) \subset g^{-1}(K) \). But \( f^{-1}(K) \) is closed (\( f \) is continuous) and \( g^{-1}(K) \) is compact (\( g \) is proper) so \( f^{-1}(K) \) is compact. This shows that \( f \) is proper. It is easy to prove that the image of a proper continuous map is closed and that a proper injective continuous map is a homeomorphism onto its image.

4 Covering dimension

4.1. An open cover of a topological space \( M \) is an indexed collection \( U = \{U_i\}_{i \in I} \) of open subsets of \( M \) such that \( M = \bigcup_{i \in I} U_i \). For each finite subset \( \sigma \) of the index set \( I \) of a cover \( U \) define the open set

\[
U_\sigma := \bigcap_{i \in \sigma} U_i.
\]

Call a nonempty finite subset \( \sigma \subset I \) a simplex if \( U_\sigma \neq \emptyset \). The set of all simplices is an abstract simplicial complex (i.e. every nonempty subset of a simplex is a simplex) called the nerve of the cover. A cover is said to have order \( \leq m \) iff every simplex has cardinality at most \( m + 1 \).

4.2. A refinement of the open cover \( U \) consists of an open cover \( V \) and a map \( V \to U : V \mapsto U_V \) such that \( V \subset U_V \) for all \( V \in V \). We say that a topological space has covering dimension \( \leq m \) iff every open cover has a refinement of order \( \leq m \).
Remark 4.3. One may distinguish between indexed covers and unindexed covers. The former is an indexed collection \( \{U_i\}_{i \in I} \) of open sets and the latter is a set \( \mathcal{U} \) of open sets. An indexed cover determines an unindexed cover via the prescription
\[
U \in \mathcal{U} \iff U = U_i \text{ for some } i \in I,
\]
and every unindexed cover arises this way. In the theory of covering dimension, unindexed covers are more convenient whereas in other theories (e.g. sheaf cohomology) indexed covers are more convenient.

Theorem 4.4. Let \( K \) be an \( m \)-dimensional simplicial complex. Then the geometric realization \(|K|\) has covering dimension \( \leq m \).

Proof. Let \( \mathcal{U} \) be a an open cover of \(|K|\). By subdivision we may assume w.l.o.g. that
\[
\mathcal{U} = \{\text{star}(v) : v \in V\}, \quad \text{star}(v) = \{x \in |K| : x(v) > 0\}
\]
where \( V \) is the set of vertices of \( K \). Choose \( x_0 \in |K| \). Let
\[
\sigma = \{v \in V : x_0(v) > 0\}, \quad \text{star}(\sigma) = \bigcap_{v \in \sigma} \text{star}(v).
\]
Clearly \( \text{star}(\sigma) \) is an open neighborhood of \( x_0 \) in \(|K|\). Moreover \( \text{star}(\sigma) \) intersects at most finitely many elements \( \text{star}(w) \) of \( \mathcal{U} \). This is because if \( \text{star}(\sigma) \) and \( \text{star}(w) \) intersect, then \( \sigma \cup \{w\} \in K \) and for each \( v \in \sigma \) there are only finitely many \( \tau \in K \) with \( v \in \tau \) by part (c) of paragraph 2.1. Note that there is no bound on the number of sets \( \text{star}(w) \) which intersect \( \text{star}(\sigma) \), e.g. there can be arbitrarily many edges \( \{v, w\} \) containing a given vertex \( v \).

4.5. Exercise. For \( (K, V) \) and \( \mathcal{U} \) as in Theorem 4.4 let
\[
\mathcal{V} = \{W_v\}_{v \in V}, \quad W_v = \{x \in |K| : x(v) > (m+2)^{-1}\}.
\]
Show that \( \mathcal{V} \) covers \(|K|\) and that it is a strict refinement of the cover \( \mathcal{U} \).

Remark 4.6. A closed subset \( A \) of a space \( M \) of covering dimension \( \leq m \) has covering dimension \( \leq m \). Extract a cover of order \( \leq m \); the restriction to \( A \) is a cover of \( A \) of order \( \leq m \). Any open cover of \( A \) has form \( \{U \cap A : U \in \mathcal{U}\} \) where each \( U \in \mathcal{U} \) is an open subset of \( A \). Adjoining the open set \( M \setminus A \) gives a cover of \( M \). The following Lemma is a little different.
Lemma 4.7. Suppose that $\mathcal{U}$ is an open cover of $M$ and $A$ is a closed subset of $M$. Assume that $A$ has covering dimension $\leq m$. Then there is an open cover $\mathcal{V}$ of $M$ such that $\mathcal{V}$ refines $\mathcal{U}$ and the restriction $\{A \cap V : V \in \mathcal{V}\}$ of $\mathcal{V}$ to $A$ has order $\leq m$.

Proof. Refine the open cover $\{U \cap A : U \in \mathcal{U}\}$ of $A$ to a cover of $A$ of order $\leq m$. Each set of this cover has form $A \cap W$ where $W$ is open in $M$ and $A \cap W \subset A \cap U$ for some $U \in \mathcal{U}$. Take $\mathcal{V}$ to be the set of all $W \cap U$ together with the set of all $U \setminus A$ where $U \in \mathcal{U}$.

Corollary 4.8. A closed subset of $\mathbb{R}^m$ has covering dimension $\leq m$.

Proof. By Remark 4.6 it suffices to show that $\mathbb{R}^m$ has covering dimension $\leq m$. This follows for Example 2.5 and Theorem 4.4.

Theorem 4.9. A manifold of dimension $m$ has covering dimension $m$.

Remark 4.10. Theorem 4.9 is an immediate consequence of Theorem 4.4 and the fact that a smooth manifold is triangulable (see [2] and [7]), but the latter is most easily proved as a consequence of the Easy Whitney Embedding Theorem 3.9 which is what we are now proving.

Lemma 4.11. Suppose that $N \subset M$ is open and that $\overline{N}$ and $M \setminus N$ both have covering dimension $\leq m$. Then $M$ has covering dimension $\leq m$.

Proof. This is a special case of Lemma 4.12; we give a separate proof as a warmup. Choose an open cover $\mathcal{U}$ of $M$. By Lemma 4.7 we may assume that the cover $\{U \cap N : U \in \mathcal{U}\}$ of $N$ has order $\leq m$. By Lemma 4.7 again there is a refinement $\mathcal{W}$ of $\mathcal{U}$ such that the cover $\{W \cap (M \setminus N) : W \in \mathcal{W}\}$ of $M \setminus N$ has order $\leq m$. Since $\mathcal{W}$ refines $\mathcal{U}$ there is a map $\mathcal{W} \to \mathcal{U}$: $W \mapsto U_W$ such that $W \subset U_W$ for $W \in \mathcal{W}$. For each $U \in \mathcal{U}$ define an open set

$$V_U = \bigcup \{W : W \in \mathcal{W}, \ U_W = U\}$$

and let $\mathcal{V} = \{V_U : U \in \mathcal{U}\}$. We show that $\mathcal{V}$ is an open cover. Choose $p \in M$. Then $p \in W$ for some $W \in \mathcal{W}$ so $p \in U_W$ so $p \in V_U \in \mathcal{V}$ where $U = U_W$. Since $V_U \subset U$ for $U \in \mathcal{U}$ we have that $\mathcal{V}$ refines $\mathcal{U}$. We show that $\mathcal{V}$ has order $\leq m$. To this end choose $V_1, \ldots, V_k \in \mathcal{V}$ and assume that $\bigcap_{i=1}^{k} V_i \neq \emptyset$; we must show that $k \leq m + 1$. We consider two cases: (1) the set $\bigcap_{i=1}^{k} V_i$ intersects $\overline{N}$ and (2) the set $\bigcap_{i=1}^{k} V_i$ intersects $M \setminus N$. Each $V_i = V_{U_i}$ where $U_i \in \mathcal{U}$. Since $V_i \subset U_i$ and the cover $\{U \cap \overline{N} : U \in \mathcal{U}\}$ of $\overline{N}$ has order $\leq m$ it follows that $k \leq m + 1$ in case (1). On the other hand, each set $V_i$ is a union of one or more sets from the cover $\mathcal{W}$ so $k \leq m + 1$ in case (2) as well.
Lemma 4.12. Suppose that $M = \bigcup_{n=1}^{\infty} B_n$ where $B_n$ is open and $\bar{B}_n \subset B_{n+1}$. Put $B_0 = \emptyset$ and assume that each closed set $\bar{B}_{n+1} \setminus B_n$ has covering dimension $\leq m$. Then $M$ has covering dimension $\leq m$.

Proof. Choose any open cover $U$ of $M$; we will construct a refinement $V$ of $U$ of order $\leq m$. By Lemma 4.11 and induction on $n$ there is a sequence $U, V_0, V_1, V_2, \ldots$ of covers of $M$, each refining the one before, such that the restriction $\{V \cap \bar{B}_n : V \in V_n\}$ of $V_n$ to $\bar{B}_n$ has order $\leq m$. We will improve the proof of Lemma 4.11 so that the sequence of refinements has the following additional properties:

(a) If $V \in V_k$ and $V \cap \bar{B}_{n-1} \neq \emptyset$, then $V \subset B_n$, and

(b) If $V \in V_n$ and $V \cap \bar{B}_{n-1} \neq \emptyset$, then $V \in V_{n+1}$.

Once we have accomplished this we complete the proof as follows. Let

$$V = \{V : \exists n \text{ such that } V \in V_k \ \forall k \geq n\}.$$ 

If $p \in \bar{B}_{n-1}$ then, since $V_n$ is a cover of $M$, we have that $p \in V$ for some $V \in V_n$. By (b) and induction, $V \in V_k$ for all $k \geq n$ so $V \in V$. Thus $V$ is a cover. Now suppose that $V_1, \ldots, V_r \in V$ and $\bigcap_{i=1}^{r} V_i \neq \emptyset$. Then all $V_i \in V_n$ for sufficiently large $n$ and, by (a) and the fact that $V_n$ refines $V_0$ we have that $V_i = V_i \cap \bar{B}_n$ for sufficiently large $n$. But the restriction of $V_n$ to $\bar{B}_n$ has order $\leq m$ so $r \leq m + 1$ as required.

It remains to construct inductively the sequence $V_0, V_1, V_2, \ldots$ of covers. Let

$$V_0 = \{V : V = U \cap (B_{k+1} \setminus \bar{B}_{k-1}) \text{ for some } U \in U \text{ and } k = 1, 2, \ldots\}.$$ 

Then $V_0$ is a cover which refines $U$. If a set $V \in V_0$ intersects $\bar{B}_{n-1}$, then $k > n$ so $V \subset B_n$; i.e. the cover $V_0$ satisfies property (a). Each cover $V_k$ constructed below refines $V_0$; it follows that $V_k$ has property (a) as well.

Now assume inductively that $V_n$ has been constructed. By hypothesis there is a refinement $W$ of $V_n$ whose restriction to $\bar{B}_{n+1} \setminus B_n$ has order $\leq m$. Then there is a map $W \to V_n : W \mapsto V_W$ with $W \subset V_W$ for $W \in W$. Let

$$C = \{V \in V_n : V \cap \bar{B}_{n+1} \neq \emptyset \text{ and } V \cap \bar{B}_{n-1} = \emptyset\}.$$ 

and for $V \in C$ let

$$V' = \bigcup\{W \in W : V_W = V\}$$

so $V' \subset V$. Define $V_{n+1}$ to consist of
(i) all sets $U \in \mathcal{V}_n$ which intersect $B_{n-1}$, together with

(ii) all sets $V'$ as above where $V \in \mathcal{C}$, together with

(iii) all sets $W \in \mathcal{W}$ which do not intersect $B_{n+1}$.

First we show that $\mathcal{V}_{n+1}$ is a cover of $M$. Choose $p \in M$, $W \in \mathcal{W}$ with $p \in W$, and $V \in \mathcal{V}_n$ with $W \subset V$. If $V \cap B_{n-1} \neq \emptyset$ then $p \in V \in \mathcal{V}_{n+1}$ by (i). Otherwise, if $V \cap B_{n+1} \neq \emptyset$, then $p \in W \subset V' \in \mathcal{V}_{n+1}$ by (ii). Otherwise, $W \cap B_{n+1} = \emptyset$ so $p \in W \in \mathcal{V}_{n+1}$ by (iii). Next we show that $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$. Sets of type (i) are in $\mathcal{V}_n$, sets $V'$ of type (ii) satisfy $V' \subset V \subset C \subset \mathcal{V}_n$, and sets of type (iii) lie in $\mathcal{W}$ which refines $\mathcal{V}_n$. Finally we show that the restriction of $\mathcal{V}_{n+1}$ to $\bar{B}_{n+1}$ has order $\leq m$. Choose $U_1, \ldots, U_r \in \mathcal{V}_{n+1}$ with $\bigcap_{j=1}^r U_j \cap B_{n+1} = \emptyset$. Then every set $U_i$ is either of type (i) or type (ii).

Choose $p \in \bigcap_{j=1}^r U_j \cap B_{n+1}$. If $U_i$ is of type (ii), then $U_i = V_i'$ for some $V_i \in \mathcal{C} \subset \mathcal{V}_n$; choose $W_i \in \mathcal{W}$ with $p \in W_i \subset V_i'$. By construction, $W_i \neq W_j$ if $U_i \neq U_j$. If $U_i$ is type (i) then $U_i \in \mathcal{V}_n$; in this case let $V_i = U_i$. If $p \in \bar{B}_n$, then $\bigcap_{j=1}^r V_j \cap B_n = \emptyset$ so $r \leq m + 1$ since the restriction of $\mathcal{V}_n$ to $\bar{B}_n$ has order $\leq m$. If $p \notin \bar{B}_n$, then no $U_i$ is a subset of $B_n$ so, by property (a), no $U_i$ is of type (i). Hence $p \in \bigcap_{i=1}^r W_i \cap B_{n+1}$ so $r \leq m + 1$ as the restriction of $\mathcal{W}$ to $\bar{B}_{n+1} \setminus B_n$ has order $\leq m$. In either case $r \leq m + 1$ as required.

Theorem 4.13. A manifold of dimension $m$ has covering dimension $m$. (In particular an open subset of $\mathbb{R}^m$ has covering dimension $m$.

Proof. Every point of $M$ lies in a rectangle, i.e. an open set $R$ whose closure $\bar{R}$ lies in the domain $U$ of a chart $(\alpha, U)$ such that $\alpha(\bar{R})$ is a product of closed bounded intervals. Then $\bar{R}$ has covering dimension $\leq m$ by triangulation. Using a countable covering by rectangles, define inductively open sets $B_n$ with compact closure $\bar{B}_n$ so that $\bar{B}_n \subset B_{n+1}$ and each $\bar{B}_n$ is covered by finitely many rectangles. By Corollary 4.11 each $B_n$ has covering dimension $\leq m$ so the theorem follows from Lemma 4.12.

Lemma 4.14. Assume that $M$ has covering dimension $\leq m$. Then any open cover $\mathcal{U}$ of $M$ has a refinement

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_m$$

such that for each $k = 0, 1, \ldots, m$ any two elements of $\mathcal{V}_k$ are disjoint.
Proof. A refinement of a refinement is a refinement so we may assume w.l.o.g. that \( \mathcal{U} \) has order \( m \). Let \( N \) be the geometric realization of the nerve of the cover \( \mathcal{U} \). A partition of unity \( \{ \theta_i \}_{i \in I} \) subordinate to the cover \( \mathcal{U} \) give rise to a continuous map \( \theta : M \to N \) and the pull back of the open star of the vertex \( i \) is the set where \( \theta_i \) is positive. Thus it is enough to prove the theorem for \( N \), i.e. we may assume w.l.o.g that \( M \) is the geometric realization of a simplex, \( \theta_i \) is the barycentric coordinate of the \( i \)th vertex, and \( \mathcal{U} \) is the cover by open stars of vertices.

For each \( k \) simplex \( \sigma \) the set

\[
V_\sigma = \{ x \in M : \theta_i(x) < \min_{j \in \sigma} \theta_j(x) \forall i \in I \setminus \sigma \}
\]

is a neighborhood of (the geometric realization of) the \( k \)-simplex \( \sigma \). Let \( \mathcal{V}_k \) be the set of all such set \( V_\sigma \); we must show that (1) \( V_\sigma \cap V_\beta = \emptyset \) if \( \sigma \) and \( \beta \) are distinct \( k \)-simplices, and (2) every \( x \in M \) is in some \( V_\sigma \). To prove (1) choose \( r \in \sigma \setminus \beta \) and \( s \in \beta \setminus \sigma \). Then \( \theta_r(x) < \theta_s(x) \) for \( x \in V_\beta \) and \( \theta_s(x) < \theta_r(x) \) for \( x \in V_\sigma \) so \( V_\sigma \cap V_\beta = \emptyset \). To prove (2) choose \( x \in M \) and let \( \sigma = \{ i \in I : \theta_i(x) > 0 \} \). Then \( x \in V_\sigma \) since \( \theta_j(x) = 0 \) for \( j \notin \sigma \).

Corollary 4.15. A manifold of dimension \( m \) can be covered by \( m+1 \) charts.

Proof. Choose a countable cover by charts. By composing each with a suitable diffeomorphism we may suppose that images of the charts are disjoint. Then apply Lemma 4.14 and “coalesce” the charts in each \( \mathcal{V}_i \) into a single chart.
5 Tubular neighborhoods

5.1. Let $M$ be a submanifold of a smooth manifold $N$. A tubular neighborhood of $M$ in $N$ is a map $\Phi : B \to M$ where $E \to M$ is a vector bundle, $B$ is an open neighborhood of the zero section $0_E$ of $E$, and $U$ is a neighborhood of $M$ in $N$ such that $\Phi$ is a diffeomorphism onto an open subset of $M$ and $\Phi(0_p) = p$ for $p \in M$ where $0_p$ is the zero of the vector space $E_p$.

Example 5.2. Let $N = \mathbb{R}^n$ and $M$ be compact. Take $E$ to be the normal bundle of $M$, i.e. $E_p = (T_pM) \perp \subset \mathbb{R}^n$. The total space of $E$ is a submanifold of $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$:

$$E = \{(p, v) \in M \times \mathbb{R}^n : v \perp T_pM\}.$$  

For $r > 0$ let

$$B_r = \{(p, v) \in E : |v| > r\}$$

be the ball bundle of radius $r$ and define $\Phi : B)r \to \mathbb{R}^n$ by

$$\Phi(p, v) = p + v.$$  

Then $D\Phi(p, 0)(\hat{p}, \hat{v}) = \hat{p} + \hat{v}$ so

$$D\Phi(p, 0) : T_{(p, 0)}E = T_pM \times E_p \to T_p\mathbb{R}^n = \mathbb{R}^n$$

is an isomorphism. Hence $\Phi$ by the implicit function theorem and compactness, $\Phi$ is a local diffeomorphism if $r > 0$ is sufficiently small. We claim that for $r$ sufficiently small, $\Phi$ is a diffeomorphism, i.e. $\Phi$ is injective. If not, there exist sequences $(p_n, v_n), (p'_n, v'_n) \in E$ with $p_n + v_n = p'_n + v'_n$ but $(p_n, v_n) \neq (p'_n, v'_n)$ and $|v_n|, |v'_n| \to 0$. Extract subsequences so that $p_n \to p$ and $p'_n \to p'$. Then $p'_n - p_n = v_n - v'_n \to 0$ so $p = p'$. But there is a neighborhood of $p$ on which $\Phi$ is a local diffeomorphism so $(p_n, v_n) = (p'_n, v'_n)$ for $n$ large enough. This is a contradiction.

Theorem 5.3. If $E \to M$ is the vector bundle corresponding to a tubular neighborhood of $M$ in $N$, then $E \to M$ is isomorphic to the normal bundle $T_MN/TN$.

Proof. At a point $0_p$ in the zero section we have a canonical isomorphism

$$T_{0_p}E = T_pM \oplus E_p$$

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determined by the identifications $M \to 0_E : p \mapsto 0_p$ and $T_0E_p = E_p$. (In fact the tangent space to a fiber at any point may be identified with the fiber itself as the fiber is a vector space.) The derivative $d\Phi(0_p) : T_0E \to T_pM$ is an isomorphism (as $\Phi$ is a diffeomorphism) and its restriction to $T_pM$ is the identity (as $\Phi(0_p) = p$). Hence it maps $E_p$ isomorphically to a complement of $T_pM$ in $T_pN$ so the composition with the projection $T_pN \to T_pN/T_pM$ is an isomorphism.

Theorem 5.4. Let $\Phi_i : B_i \to N$ be tubular neighborhoods of $M$ in $N$ each defined on an open neighborhood of the zero section of $M$ in $N$. Then there is a neighborhood $B$ of the zero section, a vector bundle automorphism $F : E \to E$ over the identity, and a smooth homotopy $\{\Psi_t : B \to N\}_{0 \leq t \leq 1}$ such that each $\Psi_t$ is a tubular neighborhood of $M$ and $\Psi_0 \circ F^{-1} = \Phi_0|B$, and $\Psi_1 = \Phi_1|B$.

Proof. It is enough to treat the special case where $N = E$, $M$ is the zero section of $E \to M$, and $\Phi_0$ is the identity. Shrink the domain $B_1$ so that it intersects each fiber in a convex neighborhood of the origin. Then define

$$\Psi_t(z) = t^{-1}\Phi_1(tz)$$

for $t > 0$ and $z \in E$. In a local trivialization $E|U = U \times \mathbb{R}^k \to U$ we have

$$\Phi_1(x, v) = (\phi(x, v), Q(x, v))$$

where $\phi(x, 0) = x$ and $Q(x, 0) = 0$ and

$$\Psi_t(x, v) = (\phi(x, tv), t^{-1}Q(x, tv)).$$

We define $F$ by continuity in $t$ so in the local trivialization

$$F(x, v) = (x, D_2Q(x, 0)v).$$

Since $\phi(x, 0) = x$ and $Q(x, 0) = 0$ we have

$$D\Psi_1(x, 0)(\dot{x}, \dot{v}) = (\dot{x} + D_2\phi(x, 0)\dot{v}, D_2Q(x, 0)\dot{v}).$$

This implies that $D_2Q(x, 0)$ is invertible as follows. Assume that $D_2Q(x, 0)\dot{v} = 0$ but that $\dot{v} \neq 0$. Then $D\Psi_1(x, 0)(\dot{x}, \dot{v}) = (0, 0)$ where $\dot{x} = -D_2\phi(x, 0)\dot{v}$ contradicting the fact that $\Psi_1$ is a diffeomorphism.
Theorem 5.5. Let $M$ be a submanifold of $N$. Then there is a tubular neighborhood of $M$ in $N$.

5.6. There are two proofs. The first is to embed $N$ in Euclidean space $\mathbb{R}^\ell$ reading $N$ for $M$ and $\mathbb{R}^\ell$ for $\mathbb{R}^n$ in example 5.2 to get a tubular neighborhood of $N$ in $\mathbb{R}^\ell$. Using the projection in the vector bundle this gives a neighborhood $U$ of $N$ in $\mathbb{R}^\ell$ and a smooth map $\pi : U \to N$ such that $\pi|N$ is the identity. Now take $E = T_M N \cap TM^\perp$ and $\Phi(p, v) = \pi(p + v)$. It is easy to check that the derivative of $\Phi$ is invertible at any point $(p, 0)$ in the zero section. Then we argue as before. The second proof uses sprays and is explained below.

5.7. By a standard trick a second order differential equation $\ddot{x} = f(x, \dot{x})$ can be viewed as a first order differential equation $\dot{v} = f(x, v)$, $\dot{x} = v$. The way to say this intrinsically is to define a second order equation on a manifold $N$ as a vector field $S$ on $T_N$ such that $(T\tau_N) \circ S = \text{id}_{T_N}$. Here $\tau_N : TN \to N$ is the projection of the tangent bundle. so $T\tau_N : T^2 N \to TN$. If $N$ is open in $\mathbb{R}^n$ then $TN = N \times \mathbb{R}^n$ and $T^2 N = N \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and

$$
\tau_{TN}(x, v, u, w) = (x, v), \quad T\tau_N(x, v, u, w) = (x, u)
$$

so $S(x, v) = (x, v, v, f(x, v))$. A curve $\gamma : \mathbb{R} \to N$ is called a base integral curve of the second order equation $S$ iff the curve $(\gamma, \dot{\gamma}) : \mathbb{R} \to TN$ is an integral curve for the vector field $S$. The equation $S$ is called a spray iff whenever $\gamma$ is a basis integral curve and $a \in \mathbb{R}$ the curve $\gamma_a(t) = \gamma(at)$ is also a base integral curve. It follows easily that in local coordinates $f(x, v)$ has the form

$$f(x, v) = \Gamma(x)v^2$$

where $\Gamma$ is a smooth map which assigns to each $x \in N$ a smooth bilinear map $\Gamma(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. (In Riemannian geometry the equation for geodesics has this form.) There is a map $\Phi : TN \to N$ such that the base integral curve satisfying the initial condition $\gamma(0), \dot{\gamma}(0)) = (p, v)$ is $t \mapsto \Phi(p, tv)$. For each $p \in N$ the map $\Phi_p : T_p N \to N$ defined by $\Phi_p(v) = \Phi(p, v)$ satisfies

$$D\Phi_p(0)v = v$$

an thus maps a neighborhood of the origin diffeomorphically onto a neighborhood of $p$ in $N$. (In Riemannian geometry this map is called the exponential map.) It follows easily that if $E$ is a complement to the tangent bundle $TM$ in $T_M N$ then the restriction of $\Phi$ to a sufficiently small neighborhood of the zero section is a tubular neighborhood.
Remark 5.8. In this section we have tacitly assumed that $M$ and $N$ are compact. This hypothesis can be removed; the proof is routine. Also it does not follow from the definition of spray that the map $\Phi$ is defined on all of $TN$ even if $N$ is compact. The tangent bundle of $N$ is not compact and we can make examples of sprays where the integral curves are not defined for all time. (However for the geodesic spray on a compact Riemannian manifold the geodesics are defined for all time.) Again, at the cost of more cumbersome notation, this defect in the above argument can easily be repaired. (In both cases, use a countable cover by compact sets and diagonalize.)

6 Connections

6.1. Throughout this section $\pi : E \to M$ denotes a smooth fiber bundle with fiber $F$, i.e. $E$, $M$, and $F$ are smooth manifolds, $\pi$ is a smooth map, and for each point $p_0 \in M$ there is a neighborhood $U$ of $p_0$ and a diffeomorphism

$$\Phi : U \times F \to \Phi(U \times F) \subset E$$

onto an open subset of $E$ such that $\pi(\Phi(p, v)) = p$ for $p \in U$. The manifold $E$ is called the total space, the manifold $M$ is called the base space, the manifold $F$ is called the fiber, and the map $\Phi$ is called a local trivialization of $\pi$. We adopt the notational convention that

$$E = \{(p, v) : p \in M, v \in E_p\}, \quad \pi(p, v) = p$$

so that for a trivial bundle $M \times F \to M : (p, v) \mapsto v$ we have $E_p = F$ for all $p \in M$. A fiber bundle isomorphism is a commutative diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{\Phi} & E_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
M_1 & \xrightarrow{\phi} & M_2
\end{array}$$

where the horizontal arrows are diffeomorphisms and the vertical arrows are fiber bundles. Thus a local trivialization is a fiber bundle isomorphism (over the identity map of $U$) from the trivial bundle $U \times F \to U$ to the restriction $E|U \to U$ where $E|U := \pi^{-1}(U)$. 

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6.2. The pull back $f^*E \to N$ of a fiber bundle $E \to M$ by a smooth map $f : N \to M$ is the fiber bundle $f^*E \to N$ where the total space is

$$f^*E := \{(q, v) : q \in N, \ v \in E_{f(q)}\}$$

and the projection is $(q, v) \mapsto q$. If $\Phi : U \times F \to E$ is a local trivialization of $E \to M$ and $V \subseteq N$ is an open set with $f(V) \subseteq U$, then the map

$$V \times F \to f^*E : (q, v) \mapsto (q, \Phi_{f(q)}(v))$$

is a local trivialization of the pull back bundle; this uniquely determines the smooth structure on $f^*E$. Here $\Phi_p : F \to E_p$ is defined by the equation $\Phi(p, v) = (p, \Phi_p(v))$.

6.3. Let $V \subseteq TE$ denote the vertical tangent bundle, i.e.

$$V_{(p,v)} = T_vE_p \subseteq T_{(p,v)}E$$

is the tangent space at $(p, v)$ to the fiber $E_p$. A horizontal bundle for the fiber bundle is a complementary subbundle $H$ in $TE$ to the vertical bundle $V$, i.e. a direct sum splitting

$$TE = H \oplus V.$$ 

A curve $\alpha : [a, b] \to E$ is called horizontal iff

$$\dot{\alpha}(t) \in H_{\alpha(t)}$$

for $t \in [a, b]$; we call a horizontal curve $\alpha : [a, b] \to E$ a horizontal lift of $\lambda := \pi \circ \alpha$.

6.4. On a trivial bundle $U \times F \to U$ where $U$ is open in $\mathbb{R}^m$ there is a smooth map which assigns to each point $(p, v) \in U \times F$ a linear map $\Gamma(p, v) : \mathbb{R}^m \to T_vF$ such that

$$H_{(p,v)} = \{(\hat{p}, \hat{v}) \in \mathbb{R}^m \times T_vF : \hat{v} = \Gamma(p, v)\hat{p}\}.$$ 

Thus a curve $\alpha = (\lambda, \xi)$ is horizontal if and only if it solves the differential equation

$$\dot{\xi} = \Gamma(\lambda, \xi)\dot{\lambda}.$$ 

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Lemma 6.5. Let $H$ be a horizontal bundle for the fiber bundle $\pi : E \to M$. Then for every curve $\lambda : (a, b) \to M$ every $c \in (a, b)$ and every $v \in E_{\lambda(c)}$ there exists an interval $(a_1, b_1) \subset (a, b)$ and a horizontal lift $\alpha$ of $\lambda|(a_1, b_1)$ with $\alpha(c) = (\lambda(c), v)$; any two such lifts agree on their common domain.

Proof. Choose a local trivialization and use the existence and uniqueness theorem for ordinary differential equations. \hfill \square

Definition 6.6. A horizontal bundle is said to have the connection property iff for every curve $\lambda : (a, b) \to M$ every $c \in (a, b)$ and every $v \in E_{\lambda(c)}$ there is a horizontal lift $\alpha$ of $\lambda$ with $\alpha(c) = (\lambda(c), v)$; a connection is a horizontal bundle with the connection property. The connection assigns to each piecewise smooth curve $\lambda : [a, b] \to M$ a diffeomorphism 

$$\Lambda(\lambda) : E_{\lambda(a)} \to E_{\lambda(b)}$$

defined by the condition that $\Lambda(\lambda)(v) = \xi(b)$ where $(\lambda, \xi)$ is the horizontal lift of $\lambda$ with the initial condition $\xi(a) = v$. The diffeomorphism $\Lambda(\lambda)$ is called parallel transport along $\lambda$.

Theorem 6.7. Parallel transport satisfies the following axioms:

(Identity) If $\lambda$ is a constant, then $\Lambda(\lambda) = \text{id}$.

(Composition) For $c \in [a, b]$ we have $\Lambda(\lambda|[c, b]) \circ \Lambda(\lambda|[a, c]) = \Lambda(\lambda)$.

(Inverse) If $\mu(t) = \lambda(a + b - t)$, then $\Lambda(\mu) = \Lambda(\lambda)^{-1}$.

(Parameterization) We have $\Lambda(\lambda \circ \sigma) = \Lambda(\lambda|[\sigma(a_1), \sigma(b_1)])$ for any smooth map $\sigma : [a_1, b_1] \to [a, b]$ with $\sigma(a_1) \leq \sigma(b_1)$.

(Smoothness) If $\psi : N \times [a, b] \to M$ is smooth, $\psi_q(t) = \psi(q, t)$, and $\Psi(q, t) = (\psi_q(t), \Lambda(\psi_q|[a, t])$ then $\Psi : N \times [a, b] \to E$ is smooth.

Theorem 6.8. Let $\pi : E \to M$ be a fiber bundle and $f : N \to M$ be a smooth map. Then

(i) There is a unique horizontal bundle $f^\# H$ for $f^* E$ such that a curve $(\mu, \xi)$ in $f^* E$ is horizontal iff and only if the image curve $(f \circ \mu, \xi)$ in $E$ is horizontal.
(ii) If $H$ has the connection property, so does $f^\# H$; in this case, the parallel transport operations are related via the formula

$$f^* \Lambda(\mu) = \Lambda(f \circ \mu)$$

for a smooth curve $\mu : [a, b] \to N$. (The fibers $E_{(f \circ \mu)(t)}$ and $(f^* E)_{\mu(t)}$ are the same.)

**Proof.** We first prove uniqueness. Let $\Phi : U \times F \to E$ be a local trivialization and for $(p, v) \in U \times F$ let $\Gamma(p, v) : \mathbb{R}^m \to T_v F$ be the linear map whose graph satisfies

$$H_{\Phi(p, v)} = d\phi(p, v)\{(\hat{p}, \hat{v}) \in \mathbb{R}^m \times T_v F : \hat{v} = \Gamma(p, v)\hat{p}\}.$$

Then a smooth curve $(\lambda, \xi) : [a, b] \to U \times F$ is mapped to a horizontal curve by $\Phi$ if and only if it satisfies the equation

$$\dot{\xi} = \Gamma(\lambda, \xi)\dot{\lambda}.$$

An open subset $V$ of $N$ such that $f(V) \subset U$ determines a local trivialization $\Psi : V \times F \to f^* E$ via $\Psi(q, v) = \Phi(f(q), v)$. Let $\Gamma'(q, v)$ represent a horizontal bundle $H'$ in this local trivialization. The equation which says that $\Psi(\mu, \xi)$ is horizontal is

$$\dot{\xi} = \Gamma'(\mu, \xi)\dot{\mu},$$

and the equation which says that $\Phi(f \circ \mu, \xi)$ is horizontal is

$$\dot{\xi} = \Gamma(f \circ \mu, \xi)df(\mu)\dot{\mu}.$$

Thus $H'$ has the property of part (i) if and only if

$$\Gamma'(q, v) = \Gamma(f(q), v)df(q).$$

This proves uniqueness. To prove existence use the formula to define $f^\# H$. Part (ii) is immediate; given $\mu : [a, b] \to N$, $c \in [a, b]$, and $v \in (f^* E)_{\mu(c)}$, the hypothesis that $H$ has the completeness property gives an $H$-horizontal lift $(f \circ \mu, \xi)$ with $\xi(c) = v$ defined on all of $[a, b]$. Then $(\mu, \xi)$ is an $f^\# H$-horizontal lift defined on all of $[a, b]$. \qed

**Theorem 6.9.** Let $\pi : E \to M$ be a fiber bundle with connection and $f, g : N \to M$ be homotopic smooth maps. Then the bundles $f^* E \to N$ and $g^* E \to N$ are isomorphic.
Proof. We prove below that every fiber bundle admits a connection. Let $h : N \times [0, 1] \to M$ be the homotopy, i.e. $f(q) = h(0, q)$ and $g(q) = h(1, q)$ for $q \in N$. For each $q \in N$ let $\lambda_q : [0, 1] \to M$ be the curve $\lambda_q(t) = h(t, q)$; then $\Lambda(\lambda_q)$ is a diffeomorphism from $(f^*E)_q = E_{f(q)}$ to $(g^*E)_q = E_{g(q)}$. By the smoothness axiom this map defines a diffeomorphism $f^*E \to g^*E$.

**Corollary 6.10.** If two curves $\lambda, \mu : [a, b] \to M$ are smoothly homotopic with endpoints fixed then the corresponding parallel transport diffeomorphisms form $E_{\lambda(a)}$ to $E_{\lambda(b)}$ are isotopic.

**Definition 6.11.** A connection is called **flat** iff the assertion of Corollary 6.10 holds with equality and not just isotopy, i.e. $\Lambda(\lambda) = \Lambda(\mu)$ whenever two curves $\lambda, \mu : [a, b] \to M$ are smoothly homotopic with endpoints fixed.

**Definition 6.12.** When $\pi : E \to M$ is a vector bundle, a connection is called **linear** if each map $\Lambda(\lambda)$ is linear. This is the case if and only if for each local trivialization $\Phi : U \times \mathbb{R}^n \to E$ then corresponding map $(p, v) \to \Gamma(p, v)$ is linear in $v$.

**Theorem 6.13.** Every vector bundle admits a linear connection.

**Proof.** Use partitions of unity.

**Corollary 6.14.** Theorem 6.9 holds for vector bundles.

**Remark 6.15.** In topology there is a notion of path lifting function which is similar to the notion of connection. The precise definition is as follows. Let $\pi : E \to M$ be a continuous map. A path lifting function is an operation which assigns to each continuous curve $\lambda : [a, b] \to M$ and each $v \in E_{\lambda(a)}$ a continuous curve $\Lambda(\lambda, v) : [a, b] \to E$ satisfying the following axioms:

(Lifting Axiom) $\Lambda(\lambda, v)(a) = v$ and $\pi \circ \Lambda(\lambda, v) = \lambda$.

(Continuity Axiom) If $\lambda : X \times [a, b] \to M$ is continuous, $\lambda_x(t) = \lambda(x, t)$, $\Lambda_x$ is the lift of $\lambda_x$, and $\Lambda : X \times [a, b] \to E$ is defined by $\Lambda(x, t) = \Lambda_x(t)$, then $\Lambda$ is continuous.

A map $\pi$ which admits such a path lifting function is called a Hurewicz fibration. It is a theorem that for a Hurewicz fibration, the composition axiom holds up to homotopy with endpoints fixed so that for each $\lambda$ the map

$$E_{\lambda(a)} \to E_{\lambda(b)} : v \mapsto \Lambda(\lambda)(v) := \Lambda(\lambda, v)(b)$$
is a homotopy equivalence. We say that a path lifting function satisfies the Strong Composition Property iff

\[ \Lambda(\lambda|[a,c]) \circ (\lambda|[c,b]) = \Lambda(\lambda,a,b) \]

for \( a \leq c \leq b \) and all continuous \( \lambda : [a,b] \to M \). It follows easily that \( \Lambda(\lambda) \) is a homeomorphism and \( \Lambda(\lambda)^{-1} = \Lambda(\lambda^{-1}) \) where \( \lambda^{-1}(t) = \lambda(b-t+a) \). It is not hard to prove (see []) that the Strong Composition Property is automatically flat, i.e. the homeomorphism \( \Lambda(\lambda) \) depends only on the homotopy class (end points fixed) of \( \lambda \). In particular, if a smooth connection is not flat, one cannot use it to lift continuous curves.

7 Transversality Theory

7.1. Let \( X \) and \( Y \) be smooth manifolds, \( W \subset Y \) a smooth submanifold, and \( f : X \to Y \) a smooth map. We say \( f \) is transverse to \( W \) at \( x \in f^{-1}(W) \), written \( f \pitchfork x W \), iff every tangent vector \( T_x f(T_x X) \) can be written as the sum of a vector tangent to \( W \) and a vector in the image of the tangent map \( T_x f : T_x X \to T_{f(x)} Y \):

\[ f \pitchfork x W \iff T_x f(T_x X) + T_{f(x)} W = T_{f(x)} Y. \]

We say \( f \) is transverse to \( W \), written \( F \pitchfork W \), iff \( f \pitchfork x W \) for all \( x \in f^{-1}(W) \):

\[ f \pitchfork W \iff f \pitchfork x W \quad \forall x \in f^{-1}(W). \]

Theorem 7.2 (Transversal Preimage). If \( f \pitchfork W \), then \( f^{-1}(W) \) is a submanifold of \( X \). The tangent space to \( f^{-1}(W) \) at a point \( x \in f^{-1}(W) \) is given by

\[ T_x f^{-1}(W) = (T_x f)^{-1}(T_{f(x)} W) \]

and the normal bundle to \( f^{-1}(W) \) in \( X \) is given by

\[ Nf^{-1}(W) = f^* NW \]

where \( NW \) is the normal bundle to \( W \) in \( Y \). In particular, codimension is preserved:

\[ \text{codim}(f^{-1}(W),X) = \text{codim}(W,Y). \]
7.3. We say that submanifolds $Z$ and $W$ of $Y$ intersect \textbf{transversally} and write $Z \pitchfork W$ iff the inclusion $Z \to Y$ is transverse to $W$. Thus for $y \in Z \cap W$ we have
\[ Z \pitchfork_y W \iff T_y Y = (T_y Z) + (T_y W) \]
and
\[ Z \pitchfork W \iff Z \pitchfork_y W \quad \forall y \in Z \cap W. \]
In this case the Transversal Preimage Theorem says that the intersection $Z \cap W$ is a submanifold of $Y$ and
\[ T_y (Z \cap W) = (T_y Z) \cap (T_y W) \]
for $y \in Z \cap W$ and
\[ \dim(Y) + \dim(Z \cap W) = \dim(Z) + \dim(W). \]

\textbf{Remark 7.4.} Note that empty intersections are transverse intersections:
\[ Z \cap W = \emptyset \implies Z \pitchfork W, \quad f^{-1}(W) = \emptyset \implies f \pitchfork W. \]
In particular, if $\dim(Z) + \dim(W) < \dim(Y)$ then $Z$ and $W$ intersect transversally if and only if they do not intersect at all. For example, if $W \subseteq Y$ is a submanifold and $\text{codim}(W, Y) > 1$ then a curve $c : \mathbb{R} \to Y$ is transverse to $W$ iff it misses $W$. This observation, together with the Transversal Density Theorem discussed below, implies that the complement of a submanifold of codimension greater than one in a a connected manifold is itself connected.

\textbf{Remark 7.5.} An important example is where $Y \to X$ is a vector bundle, $f : X \to Y$ is a section and $W$ is the (image of the) zero section. For $x \in W$ there is a natural splitting
\[ T_x Y = T_x X \oplus Y_x \]
where $Y_x$ is the fiber above $x$. In a local trivialization $f$ has the form
\[ X_0 \to X_0 \times V : x \mapsto (x, f_0(x)). \]
where $V$ is a vector space. Thus $f \pitchfork W$ if and only if the derivative $Df_0(x) : T_x X \to Y_x$ is surjective.
7.6. Now suppose that $\mathcal{A}$ is a manifold and

$$F : \mathcal{A} \times X \to Y$$

is a smooth map. For each $a \in \mathcal{A}$ define $F_a : X \to Y$ by

$$F_a(x) = F(a, x).$$

Denote by

$$\mathcal{A}_W = \{a \in \mathcal{A} : F_a \pitchfork W\}.$$

In this situation we have three fundamental theorems:

**Theorem 7.7 (Transversal Openness).** If $X$ is compact and $W$ is closed, then $\mathcal{A}_W$ is an open subset of $\mathcal{A}$.

**Theorem 7.8 (Transversal Isotopy).** Suppose that $X$ is compact and $W$ is closed. Let $\mu : [0, 1] \to \mathcal{A}_W$ be a smooth curve. Then there is a smooth isotopy $[0, 1] \to \text{diff}(X) : t \mapsto \phi_t$ such that

$$\phi_t(F^{-1}_{\mu(t)}(W)) = F^{-1}_{\mu(t)}(W).$$

**Remark 7.9.** If $\text{codim}(W, Y) = \dim(X)$ and $f \pitchfork W$, then $\dim(f^{-1}(W)) = 0$. If in addition $W$ is closed and $X$ is compact, then $f^{-1}(W)$ is finite. In this situation, the Transversal Isotopy Theorem says that the cardinality of the finite set $F_a^{-1}(W)$ is a locally constant function of $a$.

**Theorem 7.10 (Transversal Density Theorem).** If $F \pitchfork W$, then $\mathcal{A}_W$ is a residual subset of $\mathcal{A}$.

**Remark 7.11.** Recall that a subset of $\mathcal{A}$ is called **residual** iff it is countable intersection of open dense subsets. By the Baire Category Theorem a residual subset is dense. In fact, the proof shows that the complement $\mathcal{A} \setminus \mathcal{A}_W$ has measure zero (with respect to any measure that is absolutely continuous with respect to Lebesgue measure). Under the hypotheses of the Openness Theorem this implies that $\mathcal{A}$ is open dense.

**Proof of Theorem 7.10.** By hypothesis, $F^{-1}(W)$ is a smooth submanifold of $\mathcal{A} \times X$; Let $\pi : F^{-1}(W) \to \mathcal{A}$ be the projection on the first factor. We will show that $a$ is a regular value of $\pi$ if and only if $F_a \pitchfork W$. Then the result follows by Sard’s Theorem.
Assume that \( a \) is a regular value of \( \pi \). Let \( x \in F_a^{-1}(W) \) and \( y = F_a(x) \). Choose \( \hat{y} \in T_y Y \). Since \( F \cap W \) there exists \( \hat{w} \in T_y W \) and \( (\hat{a}, \hat{x}) \in T_a A \times T_x X \) with
\[
\hat{y} = \hat{w}_1 + dF(a, x)(\hat{a}, \hat{x}_1). \tag{1}
\]
Since \( a \) is a regular value of \( \pi \) there exists \( \hat{x}_2 \in T_x X \) with \( (\hat{a}, \hat{x}_2) \in T_{(a, x)} F^{-1}(W) \), i.e.
\[
\hat{w}_2 := dF(a, x)(\hat{a}, \hat{x}_2) \in T_y W. \tag{2}
\]
Subtract (2) from (1) to get
\[
\hat{y} = (\hat{w}_1 - \hat{w}_2) + dF(a, x)(\hat{x}_1 - \hat{x}_2).
\]
Since \( \hat{y} \) was arbitrary this shows that \( F \cap W \).

Assume that \( F \cap W \). Choose \((a, x) \in F_a^{-1}(W) \). Choose \( \hat{a} \in T_a A \). Let \( y = F(a, x) \) and
\[
\hat{y} = dF(a, x)(\hat{a}, 0). \tag{3}
\]
Then there exists \( \hat{w} \in T_y W \) and \( \hat{x} \in T_x X \) such that
\[
\hat{y} = \hat{w} + dF_a(x)\hat{x}. \tag{4}
\]
From (3) and (4) we get
\[
\hat{w} = dF(a, x)(\hat{a}, -\hat{x})
\]
which says that \((\hat{a}, -\hat{x}) \in T_{(a, x)} F^{-1}(W) \). Since \( \hat{a} \) was arbitrary this shows that \( a \) is a regular value of \( \pi \). \( \square \)

**Lemma 7.12 (Implicit Function Theorem).** Let \( U \) be an open set in \( \mathbb{R}^d \), \( B_r \) the ball of radius \( r \) about the origin in \( \mathbb{R}^k \), \( F : U \times B_r \to \mathbb{R}^k \) be a smooth function, and \( u_0 \in U \). Assume that \( F(u_0, 0) = 0 \) and \( D_2 F(u_0, 0) \) is invertible. Then there is a smooth function \( s : U' \to B_r \) defined in a neighborhood of \( Y' \) of \( u_0 \) in \( U \) such that for every sufficiently small \( \rho \in (0, r) \) there is a neighborhood \( U'' \) of \( u_0 \) in \( U \) with
\[
\{(u, v) \in U'' \times B_\rho : F(u, v) = 0 \} = \{(u, v) \in U'' \times B_\rho : v = S(u) \}.
\]

**Proof.** The usual version of the implicit function theorem says that there is a smooth function \( s \) such that \( F(u, s(u)) = 0 \); the present formulation is more precise. By Taylor’s formula
\[
F(u, v) = F(u, 0) + D_2 F(u, 0)v + R(u, v)
\]

and $D_2F(u,0)$ is invertible for $u = u_0$ and hence for $u$ near $u_0$. The value $s(u)$ is the unique fixed point of the map $\Gamma_u$ defined by

$$
\Gamma_u(v) = -D_2F(u,0)^{-1}(R(u,v) + F(u,0)).
$$

The fixed point is the limit $s(u) = \lim_{n \to \infty} v_n$ where $v_n = \Gamma^n_u(0)$. The uniqueness of the fixed point is assured if the Lipschitz constant $\lambda$ of $\Gamma_u | B_\rho$ is less than one for $u \in U''$; existence is assured if, in addition, we have that $|\Gamma_u(0)| < \rho/(1 - \lambda)$.

\[\Box\]

**Lemma 7.13 (Implicit Section Theorem).** Assume that $X$ is compact and $W$ is closed. Let $a_0 \in \mathcal{A}_W$, $f = F_{a_0}$, $Z = f^{-1}(W)$, and $\Phi : E \to Z$ be a tubular neighborhood of $Z$ in $X$. Then there is a neighborhood $\mathcal{A}_0$ of $a_0$ and a smooth map $S : \mathcal{A}_0 \times Z \to E$ such that for each $a \in \mathcal{A}_0$

(i) the map $S_a := S(a, \cdot)$ is a section of the vector bundle $E \to Z$,

(ii) $S_{a_0}$ is the zero section, and

(iii) $\Phi(S_a(Z)) = F^{-1}_a(W)$.

**Proof.** Equip $E$ with a fiber Riemannian metric and let $E_r$ be the open ball bundle of radius $r$. Define

$$
\Phi_a := F_a \circ \Phi : E_r \to Y, \quad \phi := \Phi_{a_0} = f \circ \Phi.
$$

Since $X \setminus \Phi(E_r)$ is compact and $F^{-1}_{a_0}(X \setminus \Phi(E_r)) \cap W = \emptyset$, we may shrink $\mathcal{A}$ and assume w.l.o.g. that $F_a(X \setminus \Phi(E_r)) \cap W = \emptyset$ for $a \in \mathcal{A}$. Hence it is enough to to construct $r > 0$, $\mathcal{A}_0$, and $S$ satisfying (i) and (ii) and the condition

(iii') $S_a(Z) = \Phi^{-1}_a(W)$.

At each point $z_0 \in Z$ there is a neighborhood $Z_0$ of $z_0$ in $Z$, a number $r > 0$, a chart $(Y_0, \beta)$ on $Y$ at $w_0 = f(z_0)$ with $\beta(Y_0) = V_1 \times V_2 \subset \mathbb{R}^\ell \times \mathbb{R}^k$ with $\beta(w_0) = (0,0)$, $\beta(W \cap Y_0) = V_1 \times 0$, and $\phi(E_r|Z_0) \subset Y_0$. Shrinking $r$ we may assume that $\Phi_a(E_r|Z_0) \subset V$ for a sufficiently near $a_0$. Then $\Phi^{-1}_a(W) = (\beta_2 \circ \Phi_a)^{-1}(0)$ where $\beta = (\beta_1, \beta_2) \in V_1 \times V_2$. Shrink $Z_0$ if necessary so that there is a local trivialization $\Theta : E|Z_0 \to Z_0 \times \mathbb{R}^k$ so that $\Theta(E_r|Z_0) = Z_0 \times B_r$ where $B_r$ is the ball of radius $r$ in $\mathbb{R}^k$. Now consider $g : \mathcal{A} \times Z_0 \times B_r \to V_2$ by

$$
g(a, z, v) = \beta_2(\Phi_a(\Theta((z,v)))).
$$
The hypothesis of transversality gives that $D_{3g}(a_0, z_0, 0)$ is invertible. Then, after shrinking $Z_0$ and $A$ if necessary, the Implicit Function Theorem gives a function $s : A \times Z_0 \to B_r$ such that

$$g^{-1}(0) = \{(a, z, v) \in A \times Z_0 \times B_r : v = s(a, z)\}$$

and hence $\Phi^{-1}_a \cap (E_r|Z_0) = S_a(Z_0)$ for $a \in A_0$ where $S_a(z) = \Theta^{-1}(z, s(a, z))$. Now cover $Z$ by finitely many such charts and shrink $A$ so that for $a \in A$, the section $S_a$ is defined in each of them. Thus for $a \in A \Phi^{-1}_a(W)$ intersects each fiber $\Phi(E_z \cap E_r)$ in a unique point and thus defines a global section satisfying (iii').

**Proof of Theorem 7.8.** Consider the set

$$E := \{(t, x) \in [0, 1] \times X : x \in F^{-1}_\mu(W)\}.$$ 

By Lemma 7.13 the projection $E \to [0, 1]$ is a fiber bundle with fiber $Z = F^{-1}_\mu(W)$. By Theorem 6.9 this bundle is trivial, i.e. there is a family $\{\psi_t : Z \to X\}_{t \in [0, 1]}$ of embeddings with $\psi_t(Z) = F^{-1}_\mu(W)$. For each $t \in [0, 1]$ define a vector field along $\psi_t$ by

$$\zeta_t(\psi_t(z)) = \frac{d}{ds} \psi_s(z) \bigg|_{s=t}.$$ 

Extend $\zeta$ to a global time dependent vector field on $X$ using partitions of unity and let $\{\phi_t : X \to X\}_{t \in [0, 1]}$ be the isotopy defined by

$$\frac{d}{ds} \phi_s(z) \bigg|_{s=t} = \zeta_t(\phi_t(z)), \quad \phi_0(z) = z.$$ 


8 Abstract Homology

In this section we summarize the homological algebra needed in the sequel. In applications the objects discussed below come with gradings which are not mentioned here.

8.1. A chain complex is a pair \((C, d)\) consisting of a vector space \(\Omega\) and a linear map \(d : C \to C\) called such that \(d^2 = 0\). A chain map \(\phi : (C, d) \to (C', d')\) of complexes is a linear map \(\phi : C \to C'\) such that \(\phi d = d' \phi\). A chain homotopy between two chain maps \(\phi_0, \phi_1 : (C, d) \to (C', d')\) is a linear map \(\Gamma : C \to C'\) such that \(\phi_1 - \phi_0 = d' \circ \Gamma \pm \Gamma \circ d\).

The quotient space \(H = H(C) = H(C, d) = \frac{\ker(d)}{\text{image}(d)}\) is called the homology of the complex. A chain map \(\phi : (C, d) \to (C', d')\) sends the kernel of \(d\) to the kernel of \(d'\) and the image of \(d\) to the image of \(d'\) and hence induces a linear map \(H(C, d) \to H(C', d')\) also denoted by \(\phi\). It is not hard to prove that two chain maps are chain homotopic if and only if they induce the same map on homology.

Remark 8.2. Because we assume that \(C\) is a vector space (and not just an abelian group) every chain complex is isomorphic to one of form

\[ C = V \times H \times V, \quad d(v, h, w) = (0, 0, v). \]

8.3. Consider a short exact sequence

\[ 0 \to A \overset{i}{\to} B \overset{p}{\to} C \to 0. \]

of vector spaces. Then each of the following naturally determines the others.

(a) an isomorphism \(\sigma : A \times C \to B\) such that \(\sigma(a, 0) = i(a)\) for \(x \in A\) and \(p\sigma(a, c) = c\) for \(a \in A\) and \(c \in C\).
(b) a left inverse $\lambda : B \to A$ for $i : A \to B$;
(c) a right inverse $\rho : C \to B$ for $p : B \to C$;
(d) a complement $F$ to the subspace $i(A) \subset B$.

These are related by
\[
\sigma(a, c) = i(a) + \rho(c), \quad \ker(\lambda) = \image(\rho) = F
\]
for $a \in A$, $c \in C$. We call $(\sigma, \lambda, \rho, F)$ a splitting of the exact sequence.

When $(A, d_A)$, $(B, d_B)$, $(C, d_C)$ are chain complexes and the maps $i$ and $p$ are chain maps the linear isomorphism $\sigma : A \times C \to B$ conjugates $d_B$ with the differential
\[
\sigma \circ d_B \circ \sigma^{-1} = \begin{bmatrix} d_A & \delta \\ 0 & d_C \end{bmatrix}
\]
on $A \times C$. Since $d_B^2 = 0$ the map $\delta : C \to A$ satisfies
\[
\delta \circ d_C + d_A \circ \delta = 0.
\]
Thus $\delta$ is a chain map (up to a sign) and induces a map $H(C) \to H(A)$ which is also denoted $\delta$.

**Theorem 8.4.** This map $\delta : H(C) \to H(A)$ is independent of the choice of the splitting used to define it. Moreover, the triangle

\[
\begin{array}{ccc}
H(A) & \xrightarrow{i} & H(B) \\
\downarrow{\delta} & & \downarrow{p} \\
H(C) & &
\end{array}
\]
is exact.

**Remark 8.5.** For a splitting $(\sigma, \lambda, \rho, F)$ of a short exact sequence
\[
0 \to (A, d_A) \to (B, d_B) \to (C, d_C) \to 0
\]
of chain complexes we have that $\sigma$ is a chain map $\iff$ $\lambda$ is a chain map $\iff$ $\rho$ is a chain map $\iff$ $F$ is $d_B$ invariant. It is not hard to prove that there is a splitting having these four equivalent properties if and only if the map $\delta : H(C) \to H(A)$ is zero.
9 Three cohomology theories

9.1. In this section $M$ is a manifold, $\Omega^k(M)$ is the space of differential $k$-forms on $M$, and $\Omega^k_c(M)$ is the subspace of $k$-forms with compact support. We often do not distinguish between a form $\gamma \in \Omega^k(M)$ and its restriction $\gamma|_U \in \Omega^k(U)$ to an open subset $U \subset M$ or between a compactly supported form $\alpha \in \Omega_c(U)$ and its extension $\alpha^M$ by zero.

9.2. Now let $\pi : E \to M$ be a vector bundle. A form $\omega \in \Omega^k(E)$ defined on the total space $E$ is said to have compact vertical support iff for every point $p \in M$ there is a trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that the support of $\omega$ intersects $\pi^{-1}(U)$ in a subset of form $\Phi^{-1}(U \times K)$ where $K$ is a compact subset of the fiber $\mathbb{R}^r$. This condition is stronger than the requirement that the support of $\omega$ intersects each fiber in a compact set. For example, let $M = \mathbb{R}$, $E = M \times \mathbb{R}$, $v(x, t) = t$, $f(x, t) = xt$ and $\theta = g(t) dt \in \Omega^1_c(\mathbb{R})$. Then $v^*\theta$ has compact vertical support, but $f^*\theta$ does not. Since $d\theta = 0$ we have also that $dv^*\theta = df^*\theta = 0$. Note that integration along the fiber $E_x = \{x\} \times \mathbb{R}$ transforms $v^*\theta$ to a constant function whereas this is false for $f^*\theta$:

$$\int v^*\theta|_{E_x} = \int_{\mathbb{R}} \theta,$$
$$\int f^*\theta|_{E_x} = \begin{cases} \int \theta & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\int \theta & \text{for } x < 0. \end{cases}$$

9.3. The subspaces $\Omega^*_c(M) \subset \Omega^*(M)$ and $\Omega^*_c(E) \subset \Omega^*(E)$ are invariant under the exterior derivative $d$ so we have three cohomology theories

$$H^k(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\text{image}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

$$H^k_c(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\text{image}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))}$$

$$H^k_{cv}(E) = \frac{\ker(d : \Omega^k_{cv}(E) \to \Omega^{k+1}_{cv}(E))}{\text{image}(d : \Omega^{k-1}_{cv}(E) \to \Omega^k_{cv}(E))}$$

9.4. Let $U, V \subset M$ be open. We have exact sequences

$$0 \to \Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0. \quad (1)$$

$$0 \to \Omega^*_c(U \cap V) \to \Omega^*_c(U) \oplus \Omega^*_c(V) \to \Omega^*_c(U \cup V) \to 0. \quad (2)$$
defined by
\[ \Omega^*(U \cup V) \rightarrow \Omega^*(U) \oplus \Omega^*(V) : \gamma \mapsto (\gamma|U, \gamma|V) \]
\[ \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V) : (\alpha, \beta) \mapsto (\alpha|U \cap V - \beta|U \cap V) \]
\[ \Omega^c_e(U \cap V) \rightarrow \Omega^c_e(U) \oplus \Omega^c_e(V) : \gamma \mapsto (-\gamma^U, \gamma^V) \]
\[ \Omega^c_e(U) \oplus \Omega^c_e(V) \rightarrow \Omega^c_e(M) : (\alpha, \beta) \mapsto \alpha^M + \beta^M \]
where the superscript denotes extension by zero. The sequence
\[ 0 \rightarrow \Omega^c_e(E|U \cup V)) \rightarrow \Omega^c_e(E|U) \oplus \Omega^c_e(E|V) \rightarrow \Omega^c_e(E|U \cap V)) \rightarrow 0 \] (3)
obtained by reading \( \pi^{-1}(U) \) and \( \pi^{-1}(V) \) for \( U \) and \( V \) in (1) and restricting to the forms of compact vertical support is also exact. The three short exact sequences give three long exact Mayer Vietoris sequences:
\[ \cdots \rightarrow H^*(U \cup V) \rightarrow H^*(U) \oplus H^*(V) \rightarrow H^*(U \cap V) \rightarrow \cdots \]
\[ \cdots \rightarrow H^c_e(U \cap V) \rightarrow H^c_e(U) \oplus H^c_e(V) \rightarrow H^c_e(U \cap V) \rightarrow \cdots \]
\[ \cdots \rightarrow H^c_e(E|U \cup V)) \rightarrow H^c_e(E|U) \oplus H^c_e(E|V) \rightarrow H^c_e(E|U \cap V)) \rightarrow \cdots . \]

9.5. We give an explicit construction of splittings for the short exact sequences (1-3) of paragraph 9.4. Let \( \{\rho_U, \rho_V\} \) be a partition of unity subordinate to the cover \( \{U, V\} \):
\[ \rho_U + \rho_V = 1, \quad \text{supp}(\rho_U) \subset U, \quad \text{supp}(\rho_V) \subset V. \]
A splitting \( \rho : \Omega^*(U \cap V) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \) of (1) in paragraph 9.4 is given by
\[ \rho(\gamma) = (\rho_V \gamma, -\rho_U \gamma). \]
The corresponding left inverse \( \lambda : \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(M) \) is
\[ \lambda(\alpha, \beta) = \rho_U \alpha + \rho_V \beta. \]
The boundary operator \( \delta = \lambda \circ d \circ \rho \) is given by
\[ \delta(\gamma) = d\rho_U \wedge \gamma + d\rho_V \wedge \gamma. \]
A splitting \( \rho_c : \Omega^*(M) \rightarrow \Omega^c_e(U) \oplus \Omega^c_e(V) \) of (2) in paragraph 9.4 is given by
\[ \rho_c(\gamma) = (\rho_U \gamma, \rho_V \gamma). \]
The corresponding left inverse \( \lambda_c : \Omega^*_c(U) \oplus \Omega^*_c(V) \to \Omega^*(U \cap V) \) is
\[
\lambda_c(\alpha, \beta) = -\rho_V \alpha + \rho_U \beta.
\]
The boundary operator \( \delta_c = \lambda_c \circ d \circ \rho_c \) is given by
\[
\delta_c(\gamma) = d \rho_V \wedge \gamma = -d \rho_U \wedge \gamma.
\]
The splitting for (3) is obtained by reading \( \pi^* \rho_U \) and \( \pi^* \rho_V \) in the above formulas for the splitting of (1).

**References**


