This note shows how to use transversality theory to prove that a generic cubic hypersurface in $\mathbb{P}^3$ contains 27 lines.

1. Let $G(k, n)$ denote the Grassmann manifold of $k$-dimensional vector subspaces of $\mathbb{C}^n$. It has dimension $k(n-k)$. This manifold has an atlas consisting of $\binom{n}{k}$ charts $(\alpha_I, U_I)$, one for every $k$ element subset $I$ of $\mathbb{N}^n = \{1, 2, \ldots, n\}$. These are defined as follows. Let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$, $\mathbb{C}^I$ denote the $k$-dimensional subspace spanned by $\{e_i : i \in I\}$, $J = \mathbb{N}^n \setminus I$, $\mathbb{C}^J$ denote the $(n-k)$-dimensional subspace spanned by $\{e_j : j \in J\}$, and $\mathbb{C}^{J \times I}$ denote the vector space of complex $(n-k) \times k$ matrices viewed as linear maps from $\mathbb{C}^I$ to $\mathbb{C}^J$. The graph

$$\text{Gr}(u) = \{z \in \mathbb{C}^n : z_J = uz_I\}$$

of an element $u \in \mathbb{C}^{J \times I}$ is an element of $G(k, n)$ and the chart $\alpha_I : U_I \to \mathbb{C}^{J \times I}$ is defined by

$$U_I = \{\lambda \in G(k, n) : \lambda \cap \mathbb{C}^J = \{0\}\}, \quad u = \alpha_I(\lambda) \iff \lambda = \text{Gr}(u).$$

2. For any complex vector space $V$ let $S^k(V)$ denote the vector space of homogeneous complex polynomials of degree $k$ on $V$. The complex dimension is given by

$$\dim_{\mathbb{C}}(S^k(V)) = \binom{n+k-1}{n}, \quad n = \dim(V).$$

3. Let $X = G(2, 4)$ be the Grassmann manifold of 2-dimensional subspaces of $\mathbb{C}^4$, i.e. the space of (projective) lines in $\mathbb{C}P^3$. Let $Y$ be the total space
of the vector bundle $Y \to X$ whose fiber over $\lambda \in X = G(2, 4)$ is the 4-dimensional vector space $Y_\lambda = S^3(\lambda)$ and let $W \subset Y$ be the (image of the) zero section of this vector bundle. Each $g \in S^3(C^4)$ determines a section $F_g : X \to Y$ of $Y \to X$ via restriction: $F_g(\lambda) = g|\lambda$. Let

$$A = \{ g \in S^3(C^4) : g^{-1}(0) \cap (Dg)^{-1}(0) = \{0\}\}.$$  

By the Homogeneous Resultant Theorem, $A$ is the complement of an algebraic variety and is therefore a connected open subset of the 15-dimensional vector space $S^3(C^4)$. The image of $A$ in $CP^{14}$ may be identified with the space of nonsingular cubic hypersurfaces in $CP^3$. Evidently

$$\lambda \in F_g^{-1}(W) \iff \lambda \subseteq g^{-1}(0),$$

i.e. the cardinality of the set $F_g^{-1}(W)$ is precisely the number of (projective) lines in the cubic hypersurface $\{g = 0\} \subseteq CP^3$.

4. The evaluation map $F : A \times X \to Y$ given by $F(g, \lambda) = F_g(\lambda)$ is a degree 4 polynomial (linear in $g$, cubic in $\lambda$) when expressed in the above local coordinates. In the local trivialization corresponding to $I = \{1, 2\}$ and $J = \{3, 4\}$ the formula for $F$ is $F(g, u) = (u, f)$ where

$$f(x_1, x_2) = g(x_1, x_2, u_3 x_2 + u_4 x_1 + u_4 x_2)$$

so that in multi index notation

$$g(x_1, x_2, x_3, x_4) = \sum_{p+q+r+s=3} g_{ijrs} x_1^p x_2^q x_3^r x_4^s, \quad f(x_1, x_2) = \sum_{i+j=3} f_{ij} x_1^i x_2^j,$$

$$f_{30} = g_{3000} + g_{2010} u_{31} + g_{2001} u_{41} + g_{1020} u_{31}^2 + g_{1011} u_{31} u_{41} + g_{1002} u_{41}^2 + g_{0030} u_{31}^3 + g_{0021} u_{31}^2 u_{41} + g_{0012} u_{31} u_{41}^2 + g_{0003} u_{41}^3$$

etc. It is easy to see that $F \cap W$: to solve $DF(g, u)(\hat{g}, \hat{u}) + (\hat{w}, 0) = (\hat{v}, \hat{f})$ we take $\hat{u} = \hat{v}, \hat{w} = 0, \hat{g}_{ij00} = \hat{f}_{ij}$, and $\hat{g}_{pqr} = 0$ if $(r, s) \neq (0, 0)$. By the Transversal Density Theorem conclude that $A_W$ dense in $A$. By the Transversal Openness Theorem $A_W$ is an open set. Below we show that $A_W$ is connected. Hence, by the Transversal Isotopy Theorem, the cardinality of $F_g^{-1}(W)$ is a constant function of $g \in A_W$.  

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5. We can reach these same conclusions by arguing as follows. Each \( g \in A \subseteq S^3(\mathbb{C}^4) \) and each two element subset \( I \subseteq N_4 \) determines a polynomial map \( f_I : \mathbb{C}^{2 \times 2} \to S^3(\mathbb{C}^2) \) as above. Define the set
\[
S_I(g) = \{ u \in \mathbb{C}^{2 \times 2} : f_I(u) = 0, \quad \det(Df_I(u)) = 0 \}.
\]
Then
\[
A_W = \{ g \in A : S_I(g) = \emptyset \forall I \}.
\]
By the Resultant Theorem the set \( A_W \) is the complement of a closed algebraic set in \( S^3(\mathbb{C}^4) \) and is hence open dense and connected provided that it is nonempty.

6. Theorem. The homogeneous polynomial
\[
g(x_1, x_2, x_3, x_4) = x_1^2 + x_2^3 + x_3^3 + x_4^3
\]
is an element of \( A_W \). The associated section \( F_g \) has exactly 27 zeros.

Proof. For each 2 element subset \( I \subseteq N_4 \) and each pair \((\mu, \nu)\) of cube roots of \(-1\) we define the 2 plane \( \lambda = \lambda(I, \mu, \nu) \) by
\[
\lambda = \{ x \in \mathbb{C}^4 : x_r = \mu x_p, \ x_s = \nu x_q \}
\]
where \( I = \{ p, q \} \), \( N_4 \setminus I = \{ r, s \} \). There are 27 such lines; each is a zero of \( F_g \) since \( g|_\lambda = 0 \).

Now take \( I = \{ 1, 2 \} \). In the corresponding coordinate system on \( \mathbb{G}(2, 4) \) the map \( \lambda \mapsto (\lambda, f) \) is given by
\[
\begin{align*}
f_{30} &= 1 + u^3_{31} + u^3_{41}, \\
f_{21} &= 3 u^2_{31} u_{32} + 3 u^2_{41} u_{42}, \\
f_{12} &= 3 u_{31} u^2_{32} + 3 u_{41} u^2_{42}, \\
f_{03} &= 1 + u^3_{32} + u^3_{42}.
\end{align*}
\]
We first show that \( f = 0 \) has no solution where all \( u_{pq} \neq 0 \). Suppose the contrary. From \( f_{21} = 0 \) and \( f_{12} = 0 \) we get that \( u_{31} u_{32} = u_{41} u_{42} \), so that \( u_{31} = m u_{41} \) and \( u_{32} = m u_{42} \) for some \( m \). From \( f_{21} = 0 \) we get \((m^3 + 1) u^2_{41} u_{42} = 0 \) so \( m^3 + 1 = 0 \) so \( u^3_{31} + u^3_{41} = 0 \) which contradicts \( f_{30} = 0 \). Hence some \( u_{pq} = 0 \). If (say) \( u_{41} = 0 \) then from \( f_{30} = 0 \) we get \( u_{31} \neq 0 \) so from \( f_{21} = 0 \) we get \( u_{32} = 0 \) and \( \lambda = \lambda(I, u_{31}, u_{42}) \) as required.

To finish the proof we must show that the \( 4 \times 4 \) matrix \( \partial f / \partial u \) is invertible. Evaluate at the point
\[
u = \begin{pmatrix} u_{31} & u_{32} \\ u_{41} & u_{42} \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}.
\]
The result is
\[
\begin{pmatrix}
3u_{31}^2 & 0 & 3u_{41}^2 & 0 \\
6u_{31}u_{32} & 3u_{31}^2 & 6u_{41}u_{42} & 3u_{41}^2 \\
3u_{32}^2 & 6u_{31}u_{32} & 3u_{42}^2 & 6u_{41}u_{42} \\
0 & 3u_{32}^2 & 0 & 3u_{42}^2
\end{pmatrix}
= \begin{pmatrix}
3\mu^2 & 0 & 0 & 0 \\
0 & 3\mu^2 & 0 & 0 \\
0 & 0 & 3\nu^2 & 0 \\
0 & 0 & 0 & 3\nu^2
\end{pmatrix}
\]
which is clearly invertible. \qed