Arrow's Theorem on Fair Elections

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1 Introduction

The fair way to decide an election between two candidates a and b is majority rule; if more than half the electorate prefer a to b, then a is elected; otherwise b is elected. Arrow's theorem asserts that no fair election procedure[1] exists for choosing from among three or more candidates. This paper gives an exposition of Arrow's theorem. I learned the ultrafilter proof given below at a mathematics-economics conference held at the Univerisity of Warwick around 1975.

2 Informal examples

To get a feeling for Arrow's theorem let us consider how some existing election procedures can lead to grossly unfair results.

One commonly used procedure is to have a second "runoff" election between the top two candidates if no candidate achieves a majority in the first election. The electorate might be confronted with three candidates a, b and c with candidates a and b extreme but opposite and c moderate. Suppose that each of the three candidates is the first choice of a third of the electorate and that all the supporters of a and b have c as their second choice. It seems clear that c is the best choice, especially if the supporters of a detest b and the supporters of b detest a. However, under the runoff procedure the electorate might well be forced to choose between a and b in the second election.

Another possible procedure would be an "instant runoff". Under this procedure each elector lists the candidates in order of preference. If no candidate receives a clear majority of first place votes, all candidates but the top two are eliminated and votes are recounted with each of the two remaining candidates getting the votes of those electors who prefer him/her to the other. This procedure has the same drawback as the straight runoff.

[1]Election procedures are sometimes called social choice functions.

3 Formulation of the Theorem

Throughout I denotes the **electorate**. The elements of this set represent the people who actually do the voting. For the formulation of the theorem we need assume nothing about this set except that it is nonempty and finite. The set of **candidates** is the set

$$N = \{1, 2, \ldots, n\}$$

consisting of the first n integers. We will assume only that $n \geq 3$.

A **state** of the electorate is a function $s : I \times N \to N$ such that for each $i \in I$ the function $s_i : N \to N$ defined by

$$s_i(a) = s(i, a)$$

is a bijection. The idea is that if the state of the electorate is s, then elector i prefers candidate a to candidate b if and only if $s(i, a) > s(i, b)$. We denote by Σ the set of all states of the electorate.

Denote by $G(S)$ the set of all bijections from S to itself. Then a state of the electorate can be viewed as a function

$$I \to G(N) : i \mapsto s_i.$$

This shows that cardinality of $|\Sigma|$ of Σ is given by

$$|\Sigma| = \ldots$$

where $|I|$ is the cardinality of I and $n = |N|$ is the cardinality of N. Evidentally, Σ is a rather large set.

An **election procedure** is a function which assigns to each state of the electorate an ordering of the candidates (the result of the election). In other words, an election procedure is a function

$$f : \Sigma \to G(N).$$

The condition $f(s)(a) > f(s)(b)$ says that the election procedure f ranks candidate a ahead of candidate b when the state of the electorate is s.

The election procedure f is said to satisfy the **unanimity condition** iff for all state s and all candidates a and b we have

$$\forall i \in I[s(i, a) > s(i, b)] \implies f(s)(a) > f(s)(b).$$

We certainly wouldn't call an election procedure fair if it failed to satisfy the unanimity condition. If the entire electorate prefers candidate a to candidate b, then surely any fair election procedure would rank a ahead of b.

For any state s and pair of candidates a and b let

$$X(s, a, b) = \{i \in I : s(i, a) > s(i, b)\}$$

denote the set of electors who prefer a to b. The election procedure f is said to satisfy the **monotonicity condition** iff for all states s and s' and all candidates a and b we have

$$X(s,a,b) \subseteq X(s',a,b) \text{ and } f(s)(a) > f(s)(b) \implies f(s')(a) > f(s')(b).$$

The monotonicity condition expresses the irrelevance of third alternatives. We certainly wouldn't call an election procedure fair if it failed to satisfy the monotonicity condition.

A **dictator** for election procedure f is an elector $i \in I$ whose preferences always coincide with the result of the election. In other words, i is a dictator for f iff for all states $s \in \Sigma$ and all candidates a and b we have

$$f(s)(a) > f(s)(b) \iff s(i,a) > s(i,b).$$

There can be at most one dictator An election procedure which has a dictator satisfies both the unanimity condition and the monotonicity condition. We would hardly call an election procedure fair if it has a dictator, but:

Arrow's Theorem: *Any election procedure which satisfies both the unanimity condition and the monotonicity condition has a dictator.*

We provide a proof of this theorem in the sequel.

4 Ultrafilters

Let I be a nonempty set. A **filter** on I is a set $\mathcal{F}$ of subsets of I satisfying the following three conditions:

1. $I \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;
2. If $\alpha \subseteq \beta \subseteq I$ and $\alpha \in \mathcal{F}$, then $\beta \in \mathcal{F}$;
3. If $\alpha \in \mathcal{F}$ and $\beta \in \mathcal{F}$, then $\alpha \cap \beta \in \mathcal{F}$.

Example 1. Let γ be any proper subset of I, that is, $\gamma \subseteq I$, $\gamma \neq \emptyset$, and $\gamma \neq I$. Then the set

$$\mathcal{F} = \{\alpha \subseteq I : \gamma \subseteq \alpha\}$$

is a filter called the **principal filter** generated by γ.

Example 2. Let I be any infinite set. Then the set $\mathcal{F}$ of cofinite subsets of I is a filter. (A subset $\alpha \subseteq I$ is called **cofinite** iff its complement $I \setminus \alpha$ is finite.) This filter is not principal.

Theorem 3. *Let $\mathcal{F}$ be a filter on I. Then the following conditions are equivalent:*

1. *$\mathcal{F}$ is a maximal filter, i.e. if $\mathcal{F}'$ is a filter and $\mathcal{F} \subseteq \mathcal{F}'$ then $\mathcal{F} = \mathcal{F}'$;*

2. *$\mathcal{F}$ is a prime filter, i.e. if $\alpha, \beta \subseteq I$ and $\alpha \cup \beta \in \mathcal{F}$ then either $\alpha \in \mathcal{F}$ or $\beta \in \mathcal{F}$;*

3. *For every $\alpha \subseteq I$ either $\alpha \in \mathcal{F}$ or $I \setminus \alpha \in \mathcal{F}$;*

4. *If $\beta \subseteq I$ and $\beta \cap \alpha \in \mathcal{F}$ for all $\alpha \in \mathcal{F}$, then $\beta \in \mathcal{F}$;*

5. *If $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is a partition of I, then there is a j (necessarily unique) such that $\alpha_j \in \mathcal{F}$.*

A filter which satisfies these equivalent conditions is called an **ultrafilter**.

Example 4. A principal ultrafilter is a principal filter which is an ultrafilter. A principal filter is an ultrafilter if and only if its generator γ consists of a single point.

Theorem 5. *Every filter extends to an ultrafilter.*

Proof. Zorn's Lemma. □

Corollary 6. *There exist nonprincipal ultrafilters on an infinite set.*

Theorem 7. *On a finite set every filter is principal.*

Proof. Induction. □

5 Proof of Arrow's Theorem

A **forcing coalition** for the election procedure f is a subset α of the electorate such that for all states $s \in \Sigma$ and all candidates a and b we have

$$\forall i \in \alpha[s(i,a) > s(i,b)] \implies f(s)(a) > f(s)(b).$$

The unanimity axiom says that the entire electorate is a forcing coalition and by definition an elector $i \in I$ is a dictator iff the singleton $\{i\}$ is a forcing coalition. Arrow's theorem is an immediate consequence of Theorem 7 and the following:

Theorem 8. *Let f be an election procedure which satisfies the unanimity and monotonicity conditions and $\mathcal{F}$ be the set of forcing coalitions for f. Assume that the set N of candidates contains at least three members. Then $\mathcal{F}$ is an ultrafilter.*