Solving the Cubic

JWR

Saturday, October 5, 2002

To solve the cubic equation

$$x^3 - 3ax + b = 0$$

set $x = u + v$ so that the equation takes the form

$$u^3 + v^3 + 3(uv - a)x + b = 0$$

and impose the condition

$$uv - a = 0$$

so that the cubic equation becomes equivalent to the two equations

$$u^3 v^3 = a^3, \qquad u^3 + v^3 = -b.$$

Thus u^3 and v^3 are roots of

$$t^2 + bt + a^3 = 0$$

so

$$u^3 = \frac{-b \pm \sqrt{b^2 - 4a^3}}{2}, \qquad v = \frac{a}{u}.$$

Since u^3 has two values it seems that u has six. However

$$v^3 = \left(\frac{a}{u}\right)^3 = \frac{2a^3}{-b \pm \sqrt{b^2 - 4a^3}} = \frac{-b \mp \sqrt{b^2 - 4a^3}}{2}$$

so as u takes three of its six values, v ranges over the other three and $x = u+v$ takes only three values.

It is amusing that (generically) the construction yields non-real values for u^3 and v^3 precisely when the roots $x = u + v$ of the original equation are real. We see this by graphing $y = f(x)$ where $f(x) = x^3 - 3ax + b$ and $f'(x) = 3(x^2 - a)$. In case $a < 0$ the derivative $f'(x)$ is always positive so there is only one real root and $b^2 - 4a^3 > 0$ so the values of u^3 and v^3 are real. In case $a > 0$ the function $f(x)$ has a local minimum at $x = \sqrt{a}$ and a local maximum at $x = -\sqrt{a}$ so there are three real roots if and only if

$$f(\sqrt{a}) < 0 < f(-\sqrt{a}). \tag{$*$}$$

But $f(\sqrt{a}) = -2a^{3/2} + b$ and $f(-\sqrt{a}) = 2a^{3/2} + b$ so condition $(*)$ is equivalent to the condition $-2a^{3/2} + b < 0 < 2a^{3/2} + b$ i.e. to

$$b^2 - 4a^3 < 0$$

which is the condition that u^3 and v^3 be non-real.