Analysis of the Watt Linkage

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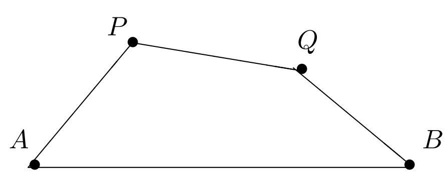
1. The **Watt linkage** or **lemniscoid** (see [1] pages 111-113)[1] consists of two fixed points A and B of distance two apart and two moveable points P and Q. Rigid bars with moveable joints are connected so that the distances $|AP|$, $|BQ|$, and $|PQ|$ are all one.



We introduce coordinates with

$$\begin{aligned} A &= (-1, 0), & B &= (1, 0), \\ P &= (x, y) = A + (X, Y), & Q &= (u, v) = B + (U, V) \end{aligned}$$

so

$$|AP|^2 = X^2 + Y^2, \qquad |BQ|^2 = U^2 + V^2, \qquad |QP|^2 = (2 + U - X)^2 + (V - Y)^2$$

so the linkage is described by the equations

$$X^2 + Y^2 = 1, \quad U^2 + V^2 = 1, \quad (2 + U - X)^2 + (V - Y)^2 = 1.$$

Using the first two equations the third simplifies to

$$6 - 4(X - U) - 2(XU + YV) = 1. \qquad (*)$$

2. Define $\alpha = \angle BAP$ and $\beta = \angle ABQ$ so that

$$\begin{aligned} x &= -1 + X = -1 + \cos\alpha, & y &= Y = \sin\alpha, \\ u &= 1 + U = 1 - \cos\beta, & v &= V = \sin\beta \end{aligned}$$

[1] Thanks to Dan Orloff for telling me about this linkage and pointing me to [1].

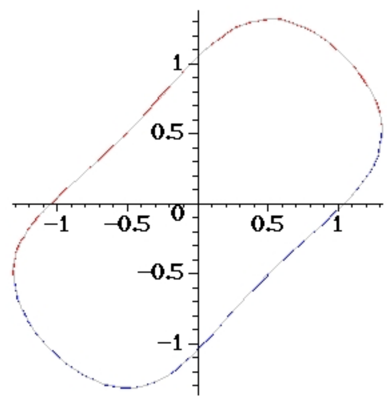


Figure 1: $6 - 4(\cos\alpha + \cos\beta) + 2\cos(\alpha + \beta) = 1$

so in the (α, β) plane the linkage has the single equation

$$6 - 4(\cos\alpha + \cos\beta) + 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = 1$$

or

$$6 - 4(\cos\alpha + \cos\beta) + 2\cos(\alpha + \beta) = 1.$$

Thus the configurations of the linkage are parameterized by the pairs (α, β) of angles satisfying the condition $|QP| = 1$. (See Figure 1.)

3. The extreme values of α occur when the points P, Q, and B are collinear and the minimum value of α is the negative of the maximum value. At the extrema the triangle PQB is isosceles with side lengths two and one so $\cos\beta = 1/4$ and $\sin\beta = \pm\sqrt{(15)}/4$. Thus the curve lies in the square

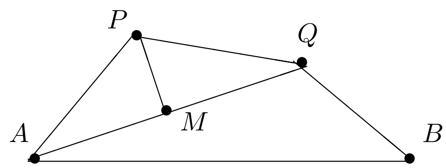
$$-\cos^{-1}(1/4) \leq \alpha, \beta \leq \cos^{-1}(1/4)$$

in the (α, β) plane. When β takes an extreme value, $\alpha = 2\beta$. Similarly reading (A, B, α, β) for (B, A, β, α).

4. We can parameterize the curve by β as follows. Write

$$\alpha = \angle BAP = \angle BAQ + \angle QAP$$

where the signs of the two angles on the right are the same if the quadrilateral $ABQP$ is convex and opposite otherwise.



Then

$$\cos\angle BAQ = \frac{2-\cos\beta}{|AQ|}, \qquad \sin\angle BAQ = \frac{\sin\beta}{|AQ|}$$

and

$$|AQ|^2 = (2-\cos\beta)^2 + (\sin\beta)^2 = 5 - 4\cos\beta.$$

Also, where M is the midpoint of AQ, we have $\angle QAP = \angle MAP$ and $|AP| = 1$ so

$$\cos\angle QAP = \frac{|AQ|}{2}, \qquad \sin\angle QAP = \pm\frac{\sqrt{4-|AQ|^2}}{2} = \pm\frac{\sqrt{4\cos\beta - 1}}{2}.$$

By the trigonometric addition formulas

$$\cos\alpha = \frac{2-\cos\beta}{2} \mp \frac{\sin\beta}{2}\sqrt{\frac{4\cos\beta-1}{5-4\cos\beta}}$$

$$\sin\alpha = \frac{\sin\beta}{2} \pm \frac{(2-\cos\beta}{2}\sqrt{\frac{4\cos\beta-1}{5-4\cos\beta}}$$

In terms of the variables X, Y, U, V we have

$$X = \frac{(2+U) - VR}{2}, \qquad Y = \frac{V + (2+U)R}{2}$$

where

$$R^2 = -\frac{4U+1}{5+4U}.$$

As a check note that by the trigonometric addition formula we have

$$\cos(\alpha+\beta) = XU - YV$$

and we recover equation $(*)$.

5. The following Maple program was used to draw the graph and to check the algebra.

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restart; with(plots): with(geometry):


QP:=(2+U-X)^2+(V-Y)^2; simplify(QP);
X:=cos(alpha); Y:=sin(alpha); U:=-cos(beta); V:=sin(beta);
PQ:=6-4*(cos(alpha)+cos(beta))+2*cos(alpha+beta);
expand(PQ)-expand(QP); # should be zero

implicitplot(PQ=1,  beta=-Pi/2..Pi/2,  alpha=-Pi/2..Pi/2,
title="6 - 4 (cos alpha + cos beta) - 2 cos(alpha-beta)=1");

bmax:=arccos(1/4); Pi/2=evalf(Pi/2); 'bmax'=evalf(bmax);

# The parameterizations:

R:=sqrt((-4*U-1)/(5+4*U));
X1:= ((2+ U)- V*R)/2;
Y1:= (V+(2+U)*R)/2;
X2:= ((2+ U)+ V*R)/2;
Y2:=(V-(2+U)*R)/2;
simplify(6-4*(X1-U)-2*(X1*U+Y1*V));
simplify(6-4*(X2-U)-2*(X2*U+Y2*V));

# As a check we plot the graph PQ=1 using the above
# parameterizations on top of the  above implicit plot.

plotsetup(jpeg,plotoutput=`watt-mwplot.jpg`);
#plotsetup(default);

P0:=implicitplot(PQ=1,  beta=-bmax..bmax,  alpha=-bmax..bmax,thickness=4,color=GREY):
P2:=plot(arcsin(Y2),beta=-bmax..bmax,color=BLUE):
P1:=plot(arcsin(Y1),beta=-bmax..bmax,color=RED):
display([P0,P1,P2],scaling=CONSTRAINED,labels=["",""]);
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References

[1] N. Rosenauer & A, H. Willis: *Kinematics of Mechanisms*, Dover Publications, 1967.