1 The moment problem

Suppose that
\[ \int x^k dF_n(x) \to \mu_k < \infty, \quad \forall k. \]
Then the following lemma guarantees the sequence of DFs is tight.

**LEM 10.1** If there is \( \phi \geq 0 \) so that \( \phi(x) \to +\infty \) as \( |x| \to \infty \) and
\[ C = \sup_n \int \phi(x) dF_n(x) < \infty, \]
then \( F_n \) is tight.

**Proof:** Note that
\[ C \geq (1 - F_n(M) + F_n(-M)) \inf_{|x| \geq M} \phi(x). \]

So every subsequential limit is a DF. Moreover, by the next lemma, the moments of the limit are given by the \( \mu_k \)'s. (Take say \( h(x) = x^k \) and \( g(x) = x^{2k} \).)

**LEM 10.2** Suppose \( g, h \) are continuous with \( g(x) > 0 \) and
\[ \frac{|h(x)|}{g(x)} \to 0, \quad \text{as} \quad |x| \to \infty. \]
If \( F_n \Rightarrow F \) and
\[ \int g(x) dF_n(x) \leq C < \infty, \]
then
\[ \int h(x) dF_n(x) \to \int h(x) dF(x). \]
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Proof: We use the method of the single probability space. Take $Y_n \rightarrow Y$ a.s. with $Y_n \sim F_n$ and $Y \sim F$. The result then follows from Theorem 1.6.8 in [D] which is the same as the above for a.s. convergence. ■

So the sequence converges weakly if there is only one DF with these moments (because every subsequence has a further subsequence converging to this unique distribution). However, in general, there can be more than one DF with the same moments.

EX 10.3 (Lognormal density) Let

$$f_0(x) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{(\log x)^2}{2}\right), \ x > 0,$$

and for $-1 \leq a \leq 1$

$$f_a(x) = f_0(x)[1 + a \sin(2\pi \log x)].$$

CLAIM 10.4

$$\int_0^\infty x^r f_0(x) \sin(2\pi \log x)dx = 0, \ r = 0, 1, 2, \ldots$$

Proof: Indeed, let $x = e^{s+r}$ so that $s = \log x - r$ and $ds = dx/x$ and

$$\int_0^\infty x^r f_0(x) \sin(2\pi \log x)dx$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp(rs + r^2) \exp\left(-\frac{(s + r)^2}{2}\right) \sin(2\pi(s + r))ds$$

$$= (2\pi)^{-1/2} \exp(r^2/2) \int_{-\infty}^{+\infty} \exp\left(-\frac{s^2}{2}\right) \sin(2\pi s)ds,$$

using that $r$ is integer. ■

2 A sufficient condition

The following condition is sufficient.

THM 10.5 If

$$\limsup_k \frac{\mu_{2k}}{2k} = r < \infty, \quad (1)$$

then there is at most one DF $F$ with

$$\mu_k = \int x^k dF(x),$$

for all positive integers $k$. 
Proof: The idea of the proof is to show that two DFs with the same moments satisfying (1) must have the same CF.

Let $F$ be any DF with moments $\mu_k$.

CLAIM 10.6 For any $\theta$

$$\phi(\theta + t) = \phi(\theta) + \sum_{m=1}^{\infty} \frac{t^m}{m!} \phi^{(m)}(\theta), \quad \forall |t| < \frac{1}{er}.$$  

The result then follows by a continuation argument. Indeed, assume there is another DF $G$ with the same moments and CF $\psi(t)$. Since $\phi(0) = \psi(0) = 1$, using induction we get that $\phi(t) = \psi(t)$ for $|t| \leq k/3r$ for all $k$. So they must be equal, and hence $F = G$. We prove the claim.

Proof: Recall:

LEM 10.7 We have

$$\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right| \leq \min \left( \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right).$$

Then

$$\left| e^{itX} \left( e^{itX} - \sum_{m=0}^{n-1} \frac{(itX)^m}{m!} \right) \right| \leq \frac{|tX|^n}{n!}.$$  

LEM 10.8 If $\mu$ has $\int |x|^n \mu(dx) < \infty$ then its CF $\phi(t)$ has continuous derivative of order $n$ given by

$$\phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx).$$

Let

$$\nu_k = \int |x|^k dF(x).$$

Taking expectations above,

$$\left| \phi(\theta + t) - \phi(\theta) - t\phi'(\theta) - \ldots - \frac{t^{n-1}}{(n-1)!} \phi^{(n-1)}(\theta) \right| \leq \frac{|t|^n}{n!} \nu_n.$$  

Cauchy-Schwarz implies that $\nu_{2k+1}^2 \leq \mu_{2k} \mu_{2k+2}$ so that

$$\limsup_k \frac{\nu_k^{1/k}}{k} \leq r.$$
(Take a further subsequence with fixed parity. If it is even, we are done. Otherwise, use that the bounding subsequence has a limit and readjust the denominator and the exponent. See Billingsley.) Hence,

\[ \nu_k \leq (r + \varepsilon)^k k^k \leq (r + \varepsilon)^k e^k k!. \]

EX 10.9 (Lognormal density continued) The moments of the lognormal are

\[ \mathbb{E}[X^n] = \mathbb{E}[e^{nZ}] = e^{n^2/2}, \]

by taking \( t = i\) in the CF of the normal.

DEF 10.10 (Carleman’s condition) A more general sufficient condition is the so-called Carleman’s condition

\[ \sum_{k=1}^{\infty} \frac{\mu_{2k}}{1/2^k} = \infty. \]

3 The method of moments

Finally, we get the following:

THM 10.11 Suppose \( \int x^k \, dF_n(x) \) has a limit \( \mu_k \) for each \( k \) with

\[ \limsup_k \frac{\mu_{2k}^{1/2k}}{2^k} = r < \infty, \]

then \( F_n \) converges weakly to the unique distribution with these moments.

References