Notes 12 : Random Walks

Math 733-734: Theory of Probability
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References: [Dur10, Section 4.1, 4.2, 4.3].

1 Random walks

**DEF 12.1** A stochastic process (SP) is a collection \( \{X_t\}_{t \in T} \) of \((E, \mathcal{E})\)-valued random variables on a triple \((\Omega, \mathcal{F}, \mathbb{P})\), where \(T\) is an arbitrary index set. For a fixed \(\omega \in \Omega\), \(\{X_t(\omega) : t \in T\}\) is called a sample path.

**EX 12.2** When \(T = \mathbb{N}\) or \(T = \mathbb{Z}_+\) we have a discrete-time SP. For instance,
- \(X_1, X_2, \ldots\) iid RVs
- \(\{S_n\}_{n \geq 1}\) where \(S_n = \sum_{i \leq n} X_i\) with \(X_i\) as above

We let \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\)

(the information known up to time \(n\)).

**DEF 12.3** A random walk (RW) on \(\mathbb{R}^d\) is an SP of the form:

\[
S_n = S_0 + \sum_{i \leq n} X_i, \quad n \geq 1
\]

where the \(X_i\)s are iid in \(\mathbb{R}^d\), independent of \(S_0\). The case \(X_i\) uniform in \([-1, +1]\) is called simple random walk (SRW).

**EX 12.4** When \(d = 1\), recall that
- **SLLN**: \(n^{-1}S_n \to \mathbb{E}[X_1]\) when \(\mathbb{E}|X_1| < +\infty\)
- **CLT**:

\[
\frac{S_n - n\mathbb{E}[X_1]}{\sqrt{n\text{Var}[X_1]}} \Rightarrow N(0, 1),
\]

when \(\mathbb{E}[X_1^2] < \infty\).
These are examples of limit theorems. Sample path properties, on the other hand, involve properties of the sequence $S_1(\omega), S_2(\omega), \ldots$. For instance, let $A \subset \mathbb{R}^d$

- $\mathbb{P}[S_n \in A \text{ for some } n \geq 1]$?
- $\mathbb{P}[S_n \in A \text{ i.o.}]$?
- $\mathbb{E}[T_A]$, where $T_A = \inf\{n \geq 1 : S_n \in A\}$?

### 1.1 Stopping times

The examples above can be expressed in terms of stopping times:

**DEF 12.5** A random variable $T : \Omega \to \mathbb{Z}_+ \equiv \{0, 1, \ldots, +\infty\}$ is called a stopping time if

$$\{T \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{Z}_+, \quad \{T = n\} \in \mathcal{F}_n, \forall n \in \mathbb{Z}_+.$$

(To see the equivalence, note

$$\{T = n\} = \{T \leq n\} \setminus \{T \leq n - 1\},$$

and

$$\{T \leq n\} = \bigcup_{i \leq n} \{T = i\}.$$)

A stopping time is a time at which one decides to stop the process. Whether or not the process is stopped at time $n$ depends only on the history up to time $n$.

**EX 12.6** Let $\{S_n\}$ be a RW and $B \in \mathcal{B}$. Then

$$T = \inf\{n \geq 1 : S_n \in B\},$$

is a stopping time. This example is also called the hitting time of $B$. (Replacing the inf with a sup (over a finite time interval say) would be a typical example of something that is not a stopping time.)

### 1.2 Wald’s First Identity

Throughout, for $X_1, X_2, \ldots \in \mathbb{R}$

$$S_n = \sum_{i=1}^{n} X_i.$$
**THM 12.7** Let $X_1, X_2, \ldots \in L^1$ be iid with $\mathbb{E}[X_1] = \mu$ and let $T \in L^1$ be a stopping time. Then

$$
\mathbb{E}[S_T] = \mathbb{E}[X_1] \mathbb{E}[T].
$$

**Proof:** Let

$$
U_T = \sum_{i=1}^T |X_i|.
$$

Observe

$$
\mathbb{E}[U_T] = \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbb{1}_{\{T=n\}} \sum_{m=1}^{n} |X_m| \right] = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \mathbb{E}[|X_m| \mathbb{1}_{\{T=n\}}] = \sum_{m=1}^{\infty} \mathbb{E}[|X_m| \mathbb{1}_{\{T\geq m\}}] = \sum_{m=1}^{\infty} \mathbb{E}[|X_m| \mathbb{1}_{\{T\leq m-1\}^c}] = \sum_{m=1}^{\infty} \mathbb{E}[|X_m| \mathbb{P}[T \geq m]] = \mathbb{E}|X_1| \mathbb{E}[T] < \infty.
$$

where we used that $X_m \perp \{T \leq m-1\} \in \mathcal{F}_{m-1}$ on the second to last line. Note that we’ve proved the theorem for nonnegative $X_i$s. This calculation also justifies using Fubini for general RVs. 

**THM 12.8** Let $X_1, X_2, \ldots \in L^2$ be iid with $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = \sigma^2$ and let $T \in L^1$ be a stopping time. Then

$$
\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T].
$$

**1.3 Application: Simple Random Walk**

Let $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = 1/2$ and $T = \inf\{n \geq 1 : S_n \notin (a, b)\}$, where $a < 0 < b$. Let $S_0 = 0$. We first argue that $\mathbb{E}T < \infty$ a.s. Since $(b-a)$ steps to the right necessarily take us out of $(a, b)$,

$$
\mathbb{P}[T > n(b-a)] \leq (1 - 2^{-(b-a)})^n,
$$
by independence of the \((b - a)\)-long stretches, so that
\[
\mathbb{E}[T] = \sum_{k \geq 0} \mathbb{P}[T > k] \leq \sum_{n} (b - a)(1 - 2^{-(b-a)})^n < +\infty,
\]
by monotonicity. In particular \(T < +\infty\) a.s.

By Wald’s First Identity,
\[
a\mathbb{P}[S_T = a] + b\mathbb{P}[S_T = b] = 0,
\]
that is
\[
\mathbb{P}[S_T = a] = \frac{b}{b - a}, \quad \mathbb{P}[S_T = b] = \frac{-a}{b - a}.
\]
In other words, letting \(T_a = \inf\{n \geq 1 : S_n = a\}\)
\[
\mathbb{P}[T_a < T_b] = \frac{b}{b - a}.
\]
By monotonicity, letting \(b \to \infty\)
\[
\mathbb{P}[T_a < \infty] \geq \mathbb{P}[T_a < T_b] \to 1.
\]
Note that this is true for every \(T_x\). In particular, we come back to where we started almost surely. This property is called recurrence. We will study recurrence more closely below.

Wald’s Second Identity tells us that
\[
\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T],
\]
where \(\sigma^2 = 1\) and
\[
\mathbb{E}[S_T^2] = \frac{b}{b - a}a^2 + \frac{-a}{b - a}b^2 = -ab,
\]
so that \(\mathbb{E}T = -ab\).

2 Recurrence of SRW

We study the recurrence of SRW on \(\mathbb{Z}^d\). Recal Stirling’s formula:
\[
n! \sim n^n e^{-n} \sqrt{2\pi n}.
\]
2.1 Strong Markov property

We will need an important property of stopping times.

**DEF 12.9** For a stopping time $T$, the $\sigma$-field $F_T$ (the information known up to time $T$) is
\[
F_T = \{ A : A \cap \{ T = n \} \in \mathcal{F}_n, \forall n \}.
\]

**THM 12.10 (Strong Markov property)** Let $X_1, X_2, \ldots$ be IID, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $T$ be a stopping with $\mathbb{P}[T < \infty] > 0$. On $\{ T < \infty \}$, $\{ X_{T+n} \}_{n \geq 1}$ is independent of $\mathcal{F}_T$ and has the same distribution as the original sequence.

**Proof:** By the Uniqueness lemma, it suffices to prove
\[
\mathbb{P}[A, T < \infty, X_{T+j} \in B_j, 1 \leq j \leq k] = \mathbb{P}[A, T < \infty] \prod_{j=1}^{k} \mathbb{P}[X_j \in B_j].
\]
for all $A \in \mathcal{F}_T$, $B_1, \ldots, B_k \in \mathcal{B}$. Then sum up over the value of $N$ and use the definition of $\mathcal{F}_T$. Indeed
\[
\mathbb{P}[A, T = n, X_{T+j} \in B_j, 1 \leq j \leq k] = \mathbb{P}[A, T = n, X_{n+j} \in B_j, 1 \leq j \leq k]
\]
\[
= \mathbb{P}[A, T = n] \prod_{j=1}^{k} \mathbb{P}[X_j \in B_j].
\]

\[\Box\]

2.2 SRW on $\mathbb{Z}$

Let $S_0 = 0$ and $T_0 = \inf\{ n > 0 : S_m = 0 \}$. We give a second proof of:

**THM 12.11 (SRW on $\mathbb{Z}$)** SRW on $\mathbb{Z}$ is recurrent.

**Proof:** First note the periodicity. So we look at $S_{2n}$. Then
\[
\mathbb{P}[S_{2n} = 0] = \binom{2n}{n} 2^{-2n}
\]
\[
\sim 2^{-2n} \frac{(2n)^{2n}}{(n^n)^2} \frac{\sqrt{2n}}{\sqrt{2\pi n}}
\]
\[
\sim \frac{1}{\sqrt{\pi n}}.
\]
So
\[ \sum_m \mathbb{P}[S_m = 0] = \infty. \]

Denote
\[ T_0^{(n)} = \inf\{ m > T_0^{(n-1)} : S_m = 0 \}. \]

By the strong Markov property \( \mathbb{P}[T_0^{(n)} < \infty] = \mathbb{P}[T_0 < \infty]^n \). Note that
\[
\sum_m \mathbb{P}[S_m = 0] = \mathbb{E} \left[ \sum_m 1\{S_m = 0\} \right]
= \mathbb{E} \left[ \sum_n 1\{T_0^{(n)} < \infty\} \right]
= \sum_n \mathbb{P}[T_0^{(n)} < \infty]
= \sum_n \mathbb{P}[T_0 < \infty]^n
= \frac{1}{1 - \mathbb{P}[T_0 < \infty]}.
\]

So \( \mathbb{P}[T_0 < \infty] = 1 \).

\[ \text{2.3 SRW on } \mathbb{Z}^2 \]

Now \( X_1 \) is in \( \mathbb{Z}^2 \) and \( \mathbb{P}[X_1 = (1, 0)] = \cdots = \mathbb{P}[X_1 = (0, -1)] = 1/4. \)

\textbf{THM 12.12 (SRW on } \mathbb{Z}^2) \) \textit{SRW on } \mathbb{Z}^2 \textit{ is recurrent.}

\textbf{Proof:} Let \( R_n = (S_n^{(1)}, S_n^{(2)}) \) where \( S_n^{(i)} \) are independent SRW on \( \mathbb{Z} \). By rotating the plane by 45 degrees, one sees that the probability to be back at \((0,0)\) in SRW on \( \mathbb{Z}^2 \) is the same as that for two independent SRW on \( \mathbb{Z} \) to be back at 0 simultaneously. Therefore,
\[
\mathbb{P}[S_{2n} = (0, 0)] = \mathbb{P}[S_{2n}^{(1)} = 0]^2 \sim \frac{1}{\pi n},
\]
whose sum diverges.

\[ \text{2.4 SRW on } \mathbb{Z}^3 \]

Now \( X_1 \) is in \( \mathbb{Z}^3 \) and \( \mathbb{P}[X_1 = (1, 0, 0)] = \cdots = \mathbb{P}[X_1 = (0, 0, -1)] = 1/6. \)

\textbf{THM 12.13 (SRW on } \mathbb{Z}^3) \) \textit{SRW on } \mathbb{Z}^3 \textit{ is transient (that is, not recurrent).}
Proof: Note, since the number of steps in opposite directions has to be equal,

\[ \mathbb{P}[S_{2n} = 0] = 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-k-j)!)^2} \]

\[ = 2^{-2n} \left( \frac{2n}{n} \right) \sum_{j,k} \frac{3^{-n} n!}{j!k!(n-k-j)!} \]

\[ \leq 2^{-2n} \left( \frac{2n}{n} \right) \max_{j,k} 3^{-n} \frac{n!}{j!k!(n-k-j)!}, \]

where we used that \( \sum_{j,k} a_{j,k}^2 \leq \max_{j,k} a_{j,k} \equiv a^* \) if \( \sum_{j,k} a_{j,k} = 1 \) and \( a_{j,k} \geq 0 \).

Note that if \( j < n/3 \) and \( k > n/3 \) then

\[ \frac{(j+1)!(k-1)!}{j!k!} = \frac{j+1}{k} \leq 1. \]

That implies that the term in the max is maximized when \( j, k, (n-k-j) \) are roughly \( n/3 \). Using Stirling

\[ \frac{n!}{j!k!(n-k-j)!} \sim \frac{n^n}{j^j k^k (n-k-j)^{n-k-j}} \sqrt{\frac{n}{jk(n-k-j)}} \frac{1}{2\pi} \sim C n^{-1} 3^n, \]

if \( j, k \) are close to \( n/3 \). Hence \( \mathbb{P}[S_{2n} = 0] \sim C n^{-3/2} \) which is summable and \( \mathbb{P}[T_0 < \infty] < 1 \). Note that it implies that \( S_n \) visits 0 only finitely many times with probability 1 as the expectation of the number of visits to 0 is \( \sum_m \mathbb{P}[S_m = 0] \) (which is then finite).

**COR 12.14** SRW on \( \mathbb{Z}^d \) with \( d > 3 \) is transient.

Proof: Let \( R_n = (S_n^1, S_n^2, S_n^3) \). Let

\[ U_m = \inf\{n > U_{m-1} : R_n \neq R_{U_{m-1}} \}. \]

Then \( R_{U_n} \) is a three-dimensional SRW. It visits \((0,0,0)\) only finitely many times a.s. and the walk is transient. Indeed \( \mathbb{P}[T_0 < +\infty] = 1 \) would imply \( \mathbb{P}[T_0^{(n)} < +\infty] = 1 \) for all \( n \), which in turn would imply that \( \mathbb{P}[S_n = 0 \text{ i.o.}] = 1 \).

3 Arcsine laws

Reference: Section 4.3 in [D].
References

