1 Random walks

**DEF 10.1** A stochastic process (SP) is a collection \( \{X_t\}_{t \in \mathcal{T}} \) of \((E, \mathcal{E})\)-valued random variables on a triple \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \mathcal{T} \) is an arbitrary index set. For a fixed \( \omega \in \Omega \), \( \{X_t(\omega) : t \in \mathcal{T}\} \) is called a sample path.

**EX 10.2** When \( \mathcal{T} = \mathbb{N} \) or \( \mathcal{T} = \mathbb{Z}_+ \) we have a discrete-time SP. For instance,
- \( X_1, X_2, \ldots \) iid RVs
- \( \{S_n\}_{n \geq 1} \) where \( S_n = \sum_{i \leq n} X_i \) with \( X_i \) as above

We let
\[
\mathcal{F}_n = \sigma(X_1, \ldots, X_n)
\]
(the information known up to time \( n \)).

**DEF 10.3** A random walk (RW) on \( \mathbb{R}^d \) is an SP of the form:
\[
S_n = S_0 + \sum_{i \leq n} X_i, \quad n \geq 1
\]
where the \( X_i \)s are iid in \( \mathbb{R}^d \), independent of \( S_0 \). The case \( X_i \) uniform in \( \{-1, +1\} \) is called simple random walk (SRW).

**EX 10.4** When \( d = 1 \), recall that
- **SLLN**: \( n^{-1} S_n \to \mathbb{E}[X_1] \) when \( \mathbb{E}|X_1| < +\infty \)
- **CLT**: 
\[
\frac{S_n - n\mathbb{E}[X_1]}{\sqrt{n\text{Var}[X_1]}} \Rightarrow \mathcal{N}(0, 1),
\]
when \( \mathbb{E}[X_1^2] < \infty \).
These are examples of limit theorems. Sample path properties, on the other hand, involve properties of the sequence $S_1(\omega), S_2(\omega), \ldots$. For instance, let $A \subset \mathbb{R}^d$

- $\mathbb{P}[S_n \in A \text{ for some } n \geq 1]$?
- $\mathbb{P}[S_n \in A \text{ i.o.}]$?
- $\mathbb{E}[T_A]$, where $T_A = \inf\{n \geq 1 : S_n \in A\}$?

### 1.1 Stopping times

The examples above can be expressed in terms of stopping times:

**DEF 10.5** A random variable $T : \Omega \to \mathbb{Z}_+ \equiv \{0, 1, \ldots, +\infty\}$ is called a stopping time if

$$\{T \leq n\} \in F_n, \forall n \in \mathbb{Z}_+,$$

or, equivalently,

$$\{T = n\} \in F_n, \forall n \in \mathbb{Z}_+.$$

*(To see the equivalence, note)*

$$\{T = n\} = \{T \leq n\} \setminus \{T \leq n - 1\},$$

and

$$\{T \leq n\} = \bigcup_{i \leq n} \{T = i\}.$$

A stopping time is a time at which one decides to stop the process. Whether or not the process is stopped at time $n$ depends only on the history up to time $n$.

**EX 10.6** Let $\{S_n\}$ be a RW and $B \in B$. Then

$$T = \inf\{n \geq 1 : S_n \in B\},$$

is a stopping time. This example is also called the hitting time of $B$. (Replacing the inf with a sup (over a finite time interval say) would be a typical example of something that is not a stopping time.)

### 1.2 Wald’s First Identity

Throughout, for $X_1, X_2, \ldots \in \mathbb{R}$

$$S_n = \sum_{i=1}^{n} X_i.$$
THM 10.7 Let $X_1, X_2, \ldots \in L^1$ be iid with $\mathbb{E}[X_1] = \mu$ and let $T \in L^1$ be a stopping time. Then

\[ \mathbb{E}[S_T] = \mathbb{E}[X_1] \mathbb{E}[T]. \]

Proof: Let

\[ U_T = \sum_{i=1}^{T} |X_i|. \]

Observe

\[ \mathbb{E}[U_T] = \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbb{1}_{\{T=n\}} \sum_{m=1}^{n} |X_m| \right] \]

\[ = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \mathbb{E}[|X_m| \mathbb{1}_{\{T=n\}}] \]

\[ = \sum_{m=1}^{\infty} \mathbb{E}[|X_m| \mathbb{1}_{\{T=m\}}] \]

\[ = \sum_{m=1}^{\infty} \mathbb{E}[|X_m| \mathbb{1}_{\{T < m\}}] \]

\[ = \sum_{m=1}^{\infty} \mathbb{E}[|X_m| \mathbb{P}[T > m - 1]] \]

\[ = \mathbb{E}[X_1] \mathbb{E}[T] < \infty. \]

where we used that $X_m \perp \{T \leq m - 1\} \in \mathcal{F}_{m-1}$ on the second to last line. Note that we’ve proved the theorem for nonnegative $X_i$s. This calculation also justifies using Fubini for general RVs.

THM 10.8 Let $X_1, X_2, \ldots \in L^2$ be iid with $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = \sigma^2$ and let $T \in L^1$ be a stopping time. Then

\[ \mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T]. \]

1.3 Application: Simple Random Walk

Let $\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = 1/2$ and $T = \inf\{n \geq 1 : S_n \notin (a, b)\}$, where $a < 0 < b$. Let $S_0 = 0$. We first argue that $\mathbb{E}T < \infty$ a.s. Since $(b - a)$ steps to the right necessarily take us out of $(a, b),

\[ \mathbb{P}[T > n(b-a)] \leq (1 - 2^{-(b-a)})^n, \]
by independence of the \((b - a)\)-long stretches, so that

\[
\mathbb{E}[T] = \sum_{k \geq 0} \mathbb{P}[T > k] \leq \sum_{n} (b - a)(1 - 2^{-(b-a)})^n < +\infty,
\]

by monotonicity. In particular \(T < +\infty\) a.s.

By Wald’s First Identity,

\[
a \mathbb{P}[S_T = a] + b \mathbb{P}[S_T = b] = 0,
\]

that is

\[
\mathbb{P}[S_T = a] = \frac{b}{b-a} \quad \text{and} \quad \mathbb{P}[S_T = b] = \frac{-a}{b-a}.
\]

In other words, letting \(T_a = \inf\{n \geq 1 : S_n = a\}\)

\[
\mathbb{P}[T_a < T_b] = \frac{b}{b-a}.
\]

By monotonicity, letting \(b \to \infty\)

\[
\mathbb{P}[T_a < \infty] \geq \mathbb{P}[T_a < T_b] \to 1.
\]

Note that this is true for every \(T_x\). In particular, we come back to where we started almost surely. This property is called recurrence. We will study recurrence more closely below.

Wald’s Second Identity tells us that

\[
\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T],
\]

where \(\sigma^2 = 1\) and

\[
\mathbb{E}[S_T^2] = \frac{b}{b-a}a^2 + \frac{-a}{b-a}b^2 = -ab,
\]

so that \(\mathbb{E}T = -ab\).

2 Recurrence of SRW

We study the recurrence of SRW on \(\mathbb{Z}^d\). Recall Stirling’s formula:

\[
n! \sim n^n e^{-n} \sqrt{2\pi n}.
\]
2.1 Strong Markov property

We will need an important property of stopping times.

**DEF 10.9** For a stopping time $T$, the $\sigma$-field $F_T$ (the information known up to time $T$) is

$$F_T = \{ A : A \cap \{ T = n \} \in F_n, \forall n \}.$$ 

**THM 10.10 (Strong Markov property)** Let $X_1, X_2, \ldots$ be IID, $F_n = \sigma(X_1, \ldots, X_n)$ and $T$ be a stopping with $P[T < \infty] > 0$. On $\{ T < \infty \}$, $\{ X_{T+n} \}_{n \geq 1}$ is independent of $F_T$ and has the same distribution as the original sequence.

**Proof:** By the Uniqueness lemma, it suffices to prove

$$P[A, T < \infty, X_{T+j} \in B_j, 1 \leq j \leq k] = P[A, T < \infty] \prod_{j=1}^{k} P[X_j \in B_j].$$

for all $A \in F_T$, $B_1, \ldots, B_k \in B$. Then sum up over the value of $N$ and use the definition of $F_T$. Indeed

$$P[A, T = n, X_{T+j} \in B_j, 1 \leq j \leq k] = P[A, T = n, X_{n+j} \in B_j, 1 \leq j \leq k]$$

$$= P[A, T = n] \prod_{j=1}^{k} P[X_j \in B_j].$$

2.2 SRW on $\mathbb{Z}$

Let $S_0 = 0$ and $T_0 = \inf\{ n > 0 : S_m = 0 \}$. We give a second proof of:

**THM 10.11 (SRW on $\mathbb{Z}$)** SRW on $\mathbb{Z}$ is recurrent.

**Proof:** First note the periodicity. So we look at $S_{2n}$. Then

$$P[S_{2n} = 0] = \binom{2n}{n} 2^{-2n}$$

$$\sim 2^{-2n} \frac{(2n)2^n \sqrt{2n}}{(n^n)^2 \sqrt{2\pi n}}$$

$$\sim \frac{1}{\sqrt{\pi n}}.$$
So
\[ \sum_m \mathbb{P}(S_m = 0) = \infty. \]

Denote
\[ T_0^{(n)} = \inf\{m > T_0^{(n-1)} : S_m = 0\}. \]

By the strong Markov property \( \mathbb{P}(T_0^{(n)} < \infty) = \mathbb{P}(T_0 < \infty)^n \). Note that
\[
\sum_m \mathbb{P}(S_m = 0) = E \left[ \sum_m 1\{S_m=0\} \right] = E \left[ \sum_n 1\{T_0^{(n)} < \infty\} \right] = \sum_n \mathbb{P}(T_0^{(n)} < \infty) = \sum_n \mathbb{P}(T_0 < \infty)^n = \frac{1}{1 - \mathbb{P}(T_0 < \infty)}.
\]

So \( \mathbb{P}(T_0 < \infty) = 1 \).

\[ \text{THM 10.12 (SRW on } \mathbb{Z}^2) \]
\[ \text{SRW on } \mathbb{Z}^2 \text{ is recurrent.} \]

\[ \text{Proof:} \] Let \( R_n = (S_n^{(1)}, S_n^{(2)}) \) where \( S_n^{(i)} \) are independent SRW on \( \mathbb{Z} \). By rotating the plane by 45 degrees, one sees that the probability to be back at \((0, 0)\) in SRW on \( \mathbb{Z}^2 \) is the same as that for two independent SRW on \( \mathbb{Z} \) to be back at 0 simultaneously. Therefore,
\[ \mathbb{P}(S_{2n} = (0, 0)) = \mathbb{P}(S_2^{(1)} = 0)^2 \sim \frac{1}{\pi n}, \]
whose sum diverges.

\[ \text{THM 10.13 (SRW on } \mathbb{Z}^3) \]
\[ \text{SRW on } \mathbb{Z}^3 \text{ is transient (that is, not recurrent).} \]
**Proof:** Note, since the number of steps in opposite directions has to be equal,

\[
\mathbb{P}[S_{2n} = 0] = 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-k-j)!)^2} \\
= 2^{-2n} \binom{2n}{n} \sum_{j,k} \left(3^{-n} \frac{n!}{j!k!(n-k-j)!}\right)^2 \\
\leq 2^{-2n} \binom{2n}{n} \max_{j,k} 3^{-n} \frac{n!}{j!k!(n-k-j)!},
\]

where we used that \(\sum_{j,k} a_{j,k}^2 \leq \max_{j,k} a_{j,k} \equiv a^*\) if \(\sum_{j,k} a_{j,k} = 1\) and \(a_{j,k} \geq 0\).

Note that if \(j < n/3\) and \(k > n/3\) then

\[
\frac{(j+1)!(k-1)!}{j!k!} = \frac{j+1}{k} \leq 1.
\]

That implies that the term in the max is maximized when \(j, k, (n-k-j)\) are roughly \(n/3\). Using Stirling

\[
\frac{n!}{j!k!(n-k-j)!} \sim \frac{n^n}{j^j k^k (n-k-j)^{n-j}} \sqrt{\frac{n}{jk(n-k-j)}} \frac{1}{2\pi} \sim C n^{-1/2},
\]

if \(j, k\) are close to \(n/3\). Hence \(\mathbb{P}[S_{2n} = 0] \sim C n^{-3/2}\) which is summable and \(\mathbb{P}[T_0 < +\infty] < 1\). Note that it implies that \(S_n\) visits 0 only finitely many times with probability 1 as the expectation of the number of visits to 0 is \(\sum_m \mathbb{P}[S_m = 0]\) (which is then finite).

**COR 10.14** SRW on \(\mathbb{Z}^d\) with \(d > 3\) is transient.

**Proof:** Let \(R_n = (S_{n,1}^1, S_{n,2}^2, S_{n,3}^3)\). Let

\[
U_m = \inf\{n > U_{m-1} : R_n \neq R_{U_{m-1}}\}.
\]

Then \(R_{U_n}\) is a three-dimensional SRW. It visits \((0,0,0)\) only finitely many times a.s. and the walk is transient. Indeed \(\mathbb{P}[T_0 < +\infty] = 1\) would imply \(\mathbb{P}[T_0^{(n)} < +\infty] = 1\) for all \(n\), which in turn would imply that \(\mathbb{P}[S_n = 0\ i.o.] = 1\).

## 3 Arcsine laws

Reference: Section 4.3 in [D].
References