Notes 6 : Weak convergence and CFs

Math 733 - Fall 2013 *Lecturer: Sebastien Roch*

References: [Dur10, Section 3.2-3.4], [Gam01, Sections IV.3, VII.5, VII.7], [Bil95, Section 26].

1 Convergence in distribution

We begin our study of a different kind of convergence.

1.1 Definition

DEF 6.1 (Convergence in distribution) *A sequence of DFs* $\{F_n\}_n$ converges in distribution *(or weakly) to a DF* F *if*

$$F_n(x) \to F(x),$$

for all points of continuity x *of* F*. This is also denoted* $F_n \Rightarrow F$*. Similarly, a sequence of RVs* $(X_n)_n$ *converges in distribution to a limit* X *if the corresponding sequence of DFs converges in distribution. We write* $X_n \Rightarrow X$.

EX 6.2 *To see why we restrict ourselves to points of continuity, consider the following example. Let* $X_n = X + 1/n$ *for some RV* X*. Then*

$$F_n(x) = \mathbb{P}[X + 1/n \leq x] = F(x - 1/n) \to F(x-).$$

EX 6.3 (Waiting for rare events) *Consider a sequence of Bernoulli trials with success probability* $0 < p < 1$*. Denoting by* X_n *the number of trials needed for a first success when* $p = 1/n$*, note that*

$$\mathbb{P}[X_n > x] = \left(1 - \frac{1}{n}\right)^{\lfloor x \rfloor}.$$

Hence, letting $Z_n = \frac{X_n}{n}$

$$\mathbb{P}\left[Z_n > z\right] = \left(1 - \frac{1}{n}\right)^{\lfloor nz \rfloor} \to e^{-z},$$

so that Z_n *converges to an exponential distribution.*

EX 6.4 (Birthday problem) *Let $(X_n)_n$ be IID uniform in $[N]$. Let*

$$T_N = \min\{n\,:\, X_n = X_m \textit{ for some } m < n\},$$

be the first time an element is picked twice. Then

$$\mathbb{P}[T_N > n] = \prod_{m=2}^{n}\left(1 - \frac{m-1}{N}\right).$$

We use the following lemma proved in homework.

LEM 6.5 *Assume*

$$\max_{1\leq j\leq n}|c_{j,n}| \to 0,$$

$$\sum_{j\leq n} c_{j,n} \to \lambda,$$

and

$$\sup_n \sum_{j\leq n}|c_{j,n}| < +\infty,$$

then

$$\prod_{j=1}^{n}(1 + c_{j,n}) \to e^{\lambda}.$$

By the previous lemma

$$\mathbb{P}\left[T_N/\sqrt{N} > x\right] \to e^{-x^2/2}, \quad x \geq 0.$$

For example

$$\mathbb{P}[T_{365} > 22] \approx 0.5.$$

1.2 Equivalent characterizations

We give equivalent characterizations of convergence in distribution. We begin with a useful fact which is sometimes called the Method of the Single Probability Space.

THM 6.6 (Method of the Single Probability Space) *If $F_n \Rightarrow F_\infty$ then there are RVs $(Y_n)_{n\leq\infty}$ (defined on a single probability space) with distribution function F_n so that $Y_n \to Y_\infty$ a.s.*

Proof:(sketch) Assume the F_n's are continuous and strictly increasing. The idea of the proof is to consider the unit interval $\Omega = (0,1)$ with Lebesgue measure and define

$$Y_n(\omega) = F_n^{-1}(\omega) \sim F_n.$$

For any $y < F_\infty^{-1}(\omega) = Y_\infty(\omega)$, since $F_\infty(y) < \omega$ we have $F_n(y) < \omega$, that is, $Y_n(\omega) = F_n^{-1}(\omega) > y$, for n large enough. Hence $\liminf_n Y_n(\omega) \geq Y_\infty(\omega)$. Similarly for the $\limsup$.

The general proof is in [D]. ∎

THM 6.7 *$X_n \Rightarrow X_\infty$ if and only if for every bounded continuous function g we have*

$$\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X_\infty)].$$

Proof: The easy direction follows from the previous theorem. Let $(Y_n)_n$ with DF F_n be s.t. that $Y_n \to Y_\infty$ a.s. By the (BDD), for any $g \in C_b(\mathbb{R})$

$$\mathbb{E}[g(X_n)] = \mathbb{E}[g(Y_n)] \to \mathbb{E}[g(Y_\infty)] = \mathbb{E}[g(X_\infty)].$$

For the other direction, let $g_{x,\varepsilon}$ be the bounded continuous function that is 1 up to x and decreases linearly to 0 between x and $x+\varepsilon$. Then

$$\limsup_n \mathbb{P}[X_n \leq x] \leq \limsup_n \mathbb{E}[g_{x,\varepsilon}(X_n)] = \mathbb{E}[g_{x,\varepsilon}(X_\infty)] \leq \mathbb{P}[X_\infty \leq x+\varepsilon].$$

Letting $\varepsilon \to 0$ gives

$$\limsup_n \mathbb{P}[X_n \leq x] \leq \mathbb{P}[X_\infty \leq x].$$

Similarly

$$\liminf_n \mathbb{P}[X_n \leq x] \geq \liminf_n \mathbb{E}[g_{x-\varepsilon,\varepsilon}(X_n)] = \mathbb{E}[g_{x-\varepsilon,\varepsilon}(X_\infty)] \geq \mathbb{P}[X_\infty \leq x-\varepsilon].$$

Letting $\varepsilon \to 0$ gives

$$\liminf_n \mathbb{P}[X_n \leq x] \geq \mathbb{P}[X_\infty < x].$$

We have equality at points of continuity. ∎

Further characterizations are given by the following.

THM 6.8 (Portmanteau Theorem) *The following statements are equivalent.*

1. $X_n \Rightarrow X_\infty$.
2. *For all open sets G,* $\liminf_n \mathbb{P}[X_n \in G] \geq \mathbb{P}[X_\infty \in G]$.

3. *For all closed sets K, $\limsup_n \mathbb{P}[X_n \in K] \leq \mathbb{P}[X_\infty \in K]$.*

4. *For all sets A with $\mathbb{P}[X_\infty \in \partial A] = 0$, $\lim_n \mathbb{P}[X_n \in A] = \mathbb{P}[X_\infty \in A]$.*

Proof: To see that 1. implies 2., take $Y_n \sim F_n$ converging a.s. Since G is open

$$\liminf_n \mathbb{1}_G(Y_n) \geq \mathbb{1}_G(Y_\infty),$$

and apply (FATOU). (The inequality is because of the boundary points.)

It is clear that 2. and 3. are equivalent by complement.

We prove that 2. and 3. imply 4. Let K be the closure of A and G, its interior. By the assumption,

$$\mathbb{P}[X_\infty \in G] = \mathbb{P}[X_\infty \in A] = \mathbb{P}[X_\infty \in K].$$

Then the result follows from 2. and 3. and

$$\mathbb{P}[X_n \in A] \leq \mathbb{P}[X_n \in K], \quad \mathbb{P}[X_n \in A] \geq \mathbb{P}[X_n \in G].$$

Finally 4. implies 1. by taking $A = (-\infty, x]$ where x is a point of continuity.

■

1.3 Further properties of weak convergence

THM 6.9 *Suppose that $\{X_n\}_n$ is defined on a single probability space and $X_n \sim F_n$. The following holds:*

1. *If $X_n \to X_\infty$ a.s., then $X_n \Rightarrow X_\infty$.*

2. *If $X_n \to_P X_\infty$, then $X_n \Rightarrow X_\infty$.*

Proof: Let $g \in C_b(\mathbb{R})$. Then $h(X_n) \to h(X_\infty)$ and

$$\mathbb{E}[h(X_n)] \to \mathbb{E}[h(X_\infty)],$$

by (BDD).

Note

$$\begin{aligned} F_n(x) &= \mathbb{P}[X_n \leq x, X_\infty \leq x + \varepsilon] + \mathbb{P}[X_n \leq x, X_\infty > x + \varepsilon] \\ &\leq F_\infty(x + \varepsilon) + \mathbb{P}[|X_n - X_\infty| > \varepsilon] \\ &\to F_\infty(x + \varepsilon). \end{aligned}$$

Similarly,

$$F_n(x) \geq F_\infty(x - \varepsilon) - \mathbb{P}[|X_n - X_\infty| > \varepsilon] \to F_\infty(x - \varepsilon).$$

Hence

$$F_\infty(x-\varepsilon) \leq \liminf_n F_n(x) \leq \limsup_n F_n(x) \leq F_\infty(x+\varepsilon).$$

If F_∞ is continuous at x, the limit is $F_\infty(x)$. ■

The opposite of the previous statements do not hold. For instance, take a sequence of independent Bernoulli trials. However we have the following.

THM 6.10 *Under the conditions of the previous theorem, if $X_n \Rightarrow c$ for a constant c. Then $X_n \to_P c$.*

Proof: Note that

$$\mathbb{P}[X_n > c+\varepsilon] \to 0, \quad \mathbb{P}[X_n \leq c-\varepsilon] \to 0.$$

Hence

$$\mathbb{P}[|X_n - c| > \varepsilon] \to 0.$$

■

Another useful result is the following, which will be proved in homework.

THM 6.11 (Converging together lemma) *If $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$, then $Z_n \Rightarrow X$.*

1.4 Selection theorem and tightness

THM 6.12 (Helly's Selection Theorem) *Let $\{F_n\}_n$ be a sequence of DFs. Then there is a subsequence $F_{n(k)}$ and a right-continuous non-decreasing function F so that*

$$\lim_k F_{n(k)}(x) = F(x),$$

at all continuity points x of F.

Proof: The proof proceeds from a diagonalization argument. Let $q_1, q_2, \ldots$ be an enumeration of the rationals. Define $m_0(i) = i, \forall i$. For each k, $F_m(q_k)$ is bounded and therefore there is a converging subsequence $\{m_k(i)\}_i$ of $\{m_{k-1}(i)\}_i$. Denote the limit by $G(q_k)$.

Now let $F_{n(k)} = F_{m_k(k)}$. By construction, for all $q \in \mathbb{Q}$

$$F_{n(k)}(q) \to G(q).$$

To define a right-continuous function, we let

$$F(x) = \inf\{G(q)\ :\ q \in \mathbb{Q},\ q > x\}.$$

Clearly, F is non-decreasing. Then,

$$\begin{aligned}\lim_{x_n \downarrow x} F(x_n) &= \inf\{G(q) \,:\, q \in \mathbb{Q},\ q > x_n \text{ for some } x_n\} \\ &= \inf\{G(q) \,:\, q \in \mathbb{Q},\ q > x\} \\ &= F(x).\end{aligned}$$

Moreover, let x be a point of continuity of F. Choose r_1, r_2, s such that $r_1 < r_2 < x < s$, r_2, s are rationals, and

$$F(x) - \varepsilon < F(r_1) \leq F(r_2) \leq F(x) \leq F(s) < F(x) + \varepsilon.$$

Note that

$$F_{n(k)}(r_2) \to G(r_2) \geq F(r_1) > F(x) - \varepsilon,$$

by definition of F, and

$$F_{n(k)}(s) \to G(s) \leq F(s) < F(x) + \varepsilon,$$

by considering $q > s$ and using the monotonicity of $F_{n(k)}$ (and therefore of G), so that for k large

$$F(x) - \varepsilon < F_{n(k)}(r_2) \leq F_{n(k)}(x) \leq F_{n(k)}(s) < F(x) + \varepsilon,$$

using the monotonicity of $F_{n(k)}$ again. ■

EX 6.13 *Let G be a DF and define for $0 < a < 1$*

$$F_n(x) = a\mathbb{1}_{x \geq n} + (1-a)G(x).$$

The mass a at n escapes to $+\infty$ and the limit is not a DF.

The following definition and theorem resolves this question.

DEF 6.14 (Tightness) *A sequence of DFs $\{F_n\}_n$ is* tight *if for all $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that*

$$\limsup_n 1 - F_n(M_\varepsilon) + F_n(-M_\varepsilon) \leq \varepsilon.$$

THM 6.15 *Every (weak) subsequential limit of a sequence of DFs $\{F_n\}_n$ is a DF (up to discontinuity points) if and only if $\{F_n\}_n$ is tight. (More formally, tightness is a necessary and sufficient condition that for every subsequence $\{F_{n(k)}\}_k$ there exists a further subsequence $\{F_{n(k_j)}\}_j$ and a DF F such that $F_{n(k_j)} \Rightarrow F$.)*

Proof: Assume tightness and let F be the limit of $\{F_{n(k)}\}$. Since F is non-decreasing it has at most countably many discontinuities. (Indeed the intervals $(F(x-), F(x+))$, $x \in \mathbb{R}$ are disjoint and each one that is nonempty contains a distinct rational number.) Let $r < -M_\varepsilon$ and $s > M_\varepsilon$ be points of continuity of F. Then

$$\begin{aligned} 1 - F(s) + F(r) &= \lim_k 1 - F_{n(k)}(s) + F_{n(k)}(r) \\ &\leq \limsup_n 1 - F_n(M_\varepsilon) + F_n(-M_\varepsilon) \\ &\leq \varepsilon, \end{aligned}$$

by monotonicity of F_n and the definition of $\limsup$. Hence

$$\limsup_{x \to +\infty} 1 - F(x) - F(-x) \leq \varepsilon,$$

and the result follows from the arbitrariness of ε, and the fact that F is non-negative and bounded by 1.

For the other direction, assume $\{F_n\}_n$ is not tight. Hence there is a subsequence $n(k)$ such that

$$1 - F_{n(k)}(k) + F_{n(k)}(-k) \geq \varepsilon, \quad \forall k.$$

Pass to a further subsequence $n(k_j)$ converging to a limit F, by Helly. Consider points of continuity $r < 0 < s$ of F. Then

$$\begin{aligned} 1 - F(s) + F(r) &= \lim_j 1 - F_{n(k_j)}(s) + F_{n(k_j)}(r) \\ &\geq \liminf_j 1 - F_{n(k_j)}(k_j) + F_{n(k_j)}(-k_j) \\ &\geq \varepsilon. \end{aligned}$$

The result follows from the arbitrariness of r, s. That is, there exists a weak subsequential limit that is not a DF. ∎

2 Characteristic functions

A fundamental tool to study weak convergence is the following.

2.1 Definition

DEF 6.16 (Characteristic function) *The* characteristic function *of a RV X is*

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)],$$

where the second equality is a definition and the expectations exist because they are bounded.

The CF is the probabilistic equivalent of the Fourier transform.

EX 6.17 (Gaussian distribution) *Let $X \sim \mathrm{N}(0,1)$. To compute the CF recall first that*

$$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} \mathrm{d}x = 1.$$

We want to compute

$$\phi_X(t) = \frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} e^{itx} \mathrm{d}x = \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int e^{-(x-it)^2/2} \mathrm{d}x.$$

Then, let C_R be the contour consisting of the rectangle with corners $\pm R$ and $\pm R - it$. As a composition of exponential and polynomial functions, $e^{-z^2/2}$ is holomorphic and Cauchy's integral formula gives

$$\begin{aligned} 0 &= \oint_{C_R} e^{-z^2/2} \mathrm{d}z \\ &= \int_{-R}^{R} e^{-x^2/2} \mathrm{d}x - \int_{-R}^{R} e^{-(x-it)^2/2} \mathrm{d}x \\ &\quad + \int_0^t e^{-(R-iy)^2/2} \mathrm{d}y - \int_0^t e^{-(-R-iy)^2/2} \mathrm{d}y, \end{aligned}$$

Note that

$$\left| \int_0^t e^{-(R-iy)^2/2} \mathrm{d}y \right| \leq \int_0^t \left| e^{-(R-iy)^2/2} \right| \mathrm{d}y = e^{-R^2/2} \int_0^t e^{y^2/2} \mathrm{d}y \to 0,$$

as $R \to \infty$. Taking $R \to \infty$ above gives

$$\phi_X(t) = e^{-t^2/2}.$$

There is a different proof in [D].

The main reason why CFs are useful is the following.

THM 6.18 *If X_1 and X_2 are independent then*

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t).$$

Proof: By independence (take real and imaginary parts inside the expectation on the LHS)

$$\mathbb{E}[e^{it(X_1+X_2)}] = \mathbb{E}[e^{itX_1}]\mathbb{E}[e^{itX_2}].$$

■

EX 6.19 (Continued) *If* $X_1, X_2 \sim \mathrm{N}(0,1)$ *are independent then*

$$\phi_{\frac{X_1+X_2}{\sqrt{2}}}(t) = \phi_{X_1+X_2}(t/\sqrt{2}) = \phi_{X_1}(t/\sqrt{2})\phi_{X_2}(t/\sqrt{2}) = e^{-t^2/4}e^{-t^2/4} = e^{-t^2/2}.$$

We will show below that CFs characterize the corresponding distribution so the previous example shows that the sum of independent Gaussians is Gaussian. More generally, we will use CFs to study sums of independent RVs by considering the product of their CFs.

We record a few properties of the CF.

THM 6.20 *If* ϕ *is the CF of a RV* X *then*

1. $\phi(0) = 1$.
2. $\phi(-t) = \overline{\phi(t)}$.
3. $|\phi(t)| \leq \mathbb{E}|e^{itX}| \leq 1$.
4. $|\phi(t+h) - \phi(t)| \leq \mathbb{E}|e^{ihX} - 1|$ *so* ϕ *is uniformly continuous.*
5. $\mathbb{E}[e^{it(aX+b)}] = e^{itb}\phi(at)$.

2.2 Lévy's Inversion Formula

THM 6.21 (Inversion Formula) *Let*

$$\phi(t) = \int e^{itx}\mu(\mathrm{d}x),$$

where μ *is a probability measure. Then for all* $a < b$

$$\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)\mathrm{d}t = \frac{1}{2}\mu(\{a\}) + \mu((a,b)) + \frac{1}{2}\mu(\{b\}).$$

(Moreover, if

$$\int |\phi(t)|\mathrm{d}t < +\infty,$$

then μ *has a bounded continuous density* f *with*

$$f(x) = \frac{1}{2\pi}\int e^{-itx}\phi(t)\mathrm{d}t.)$$

Proof: We will need the following auxiliary function

$$S(U) = \int_0^U \frac{\sin x}{x}\mathrm{d}x.$$

Note that the integral close to zero is bounded because $\sin x = x + o(x)$. On the other hand, the positive part and the negative part both diverge so the infinite integral does not exist. However, as proved in an exercise in [D]:

LEM 6.22 *We have*

$$S(U) \to \frac{\pi}{2},$$

as $U \to +\infty$.

Proof: By Cauchy's integral formula

$$0 = \oint_{D_{\varepsilon,U}} \frac{e^{iz}}{z}\mathrm{d}z,$$

where $D_{\varepsilon,U}$ is the boundary of the region between the two upper-half circles of radii ε and U. Along the axis, the real part of the integral vanishes by symmetry and the imaginary part converges to $2iS(U)$ as $\varepsilon \to 0$. For the integral over the small half-circle, write $\frac{e^{iz}}{z} = \frac{1}{z} + f(z)$ where f is analytic (and therefore continuous), and use $\frac{\mathrm{d}z}{z} = i\mathrm{d}\theta$ for the first integral and the ML-estimate for the second one (that is, that the norm of the integral is less than the maximum of f multiplied by the path length) to obtain that the latter integral is $-i\pi$. Plugging above gives the result if we can show that the integral over the large half-circle goes to 0 as $U \to +\infty$: note that $|z| = U$, $|e^{iz}| = |e^{ix-y}| = e^{-U\sin\theta}$, and $|\mathrm{d}z| = U\mathrm{d}\theta$; moreover by concavity $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$; and integrating gives the result. (The latter argument is known as Jordan's lemma.) ■

Note that for $u, v \in \mathbb{R}$

$$|e^{iu} - e^{iv}| \leq |u - v|,$$

because if $u < v$

$$\left|\int_u^v ie^{it}\mathrm{d}t\right| \leq \int_u^v \mathrm{d}t.$$

So by Fubini for $0 < T < \infty$

$$\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\phi(t)\mathrm{d}t = \frac{1}{2\pi}\int\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\mathrm{d}t\mu(\mathrm{d}x).$$

By the evenness of the cosine and the oddness of the sine,

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\mathrm{d}t = \frac{\mathrm{sgn}(x-a)S(|x-a|T)-\mathrm{sgn}(x-b)S(|x-b|T)}{\pi}$$

$$\to \begin{cases} 0 & \text{if } x < a \text{ or } x > b \\ \frac{1}{2} & \text{if } x = a \text{ or } x = b \\ 1 & \text{if } a < x < b, \end{cases}$$

as $T \to +\infty$. That gives the first result.

(The second result follows from Fubini and (DOM), see [D].) ■

COR 6.23 *If DFs F and G have the same CF, then they are equal.*

Proof: For any $x \in \mathbb{R}$, there is a sequence of (joint) continuity points above x converging to it and there is a sequence of (joint) continuity points below x converging to $-\infty$. Apply the previous result. ■

EX 6.24 (Continued) *Hence $(X_1 + X_2)/\sqrt{2}$ is $\mathrm{N}(0, 1)$.*

2.3 Moments and derivatives

The behavior of ϕ around 0 contains information about the tail/moments of μ:

THM 6.25 *We have*

$$\left|\mathbb{E}\left[e^{itX}\right] - \sum_{m=0}^{n}\frac{\mathbb{E}[(itX)^m]}{m!}\right| \leq \mathbb{E}\left[\min\{|tX|^{n+1}, 2|tX|^n\}\right].$$

In particular, if $\mathbb{E}[X^2] < +\infty$ then

$$\phi_X(t) = 1 + it\mathbb{E}[X] - \frac{t^2}{2}\mathbb{E}[X^2] + o(t^2).$$

Proof: This follows from the following deterministic statement proved in [D].

LEM 6.26 *We have*

$$\left|e^{ix} - \sum_{m=0}^{n}\frac{(ix)^m}{m!}\right| \leq \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right).$$

■

2.4 Continuity theorem

The connection between CFs and weak convergence is provided by the following theorem.

THM 6.27 (Lévy's Continuity Theorem) *Let μ_n, $1 \leq n \leq \infty$ be PMs with CFs ϕ_n.*

1. *If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all t.*
2. *If $\phi_n(t)$ converges pointwise to a limit $\phi(t)$ that is continuous at 0 then the associated sequence of PMs μ_n is tight and converges weakly to the measure μ with CF ϕ.*

Proof:

1. Note that e^{itx} is continuous and bounded. Therefore, if $\mu_n \Rightarrow \mu_\infty$, then $\phi_n(t) \to \phi_\infty(t)$.
2. Assume we proved tightness. Denote the DF of μ_n by F_n. Then each subsequence of F_n has a further subsequence converging to a DF F. By assumption, F must have CF ϕ and therefore every subsequential limit is equal. Let g be bounded and continuous, then every subsequence of $y_n = \int g(x)\mathrm{d}F_n(x)$ has a further subsequence converging to $y = \int g(x)\mathrm{d}F(x)$, so the whole sequence converging to that limit. (O.w. there is an open set G with $y \in G$ and a subsequence $y_{n(m)}$ such that $y_{n(m)} \notin G$. But then no subsequence of $y_{n(m)}$ can converge to y. The previous result also follows from the metrizability of weak convergence.)

   It remains to prove tightness. We begin with the following lemma of independent interest.

   LEM 6.28 *For $u > 0$, we have*

   $$\frac{1}{u}\int_{-u}^{u}(1-\phi(t))\mathrm{d}t \geq \mu\{x\,:\,|x| > 2/u\}.$$

   In other words, the behavior of ϕ close to 0 is related to the tail of μ.

   Proof: We bound the integral by using Fubini's theorem. Note that the integral of the absolute value is at most 4. Note that

   $$\frac{1}{u}\int_{-u}^{u} 1 - e^{itx}\mathrm{d}t = 2\left(1 - \frac{\sin ux}{ux}\right),$$

by the oddness of sine. (Clearly when u is small the RHS is small unless x is big.) Note also that $|\sin x| \leq |x|$ so the expression in parenthesis is ≥ 0. Discarding the integral over $(-2/u, 2/u)$ (which corresponds anyway to small values of the integrand),

$$\begin{aligned}\frac{1}{u}\int_{-u}^{u}(1-\phi(t))\mathrm{d}t &= 2\int\left(1-\frac{\sin ux}{ux}\right)\mu(\mathrm{d}x)\\ &\geq 2\int_{|x|\geq 2/u}\left(1-\frac{1}{|ux|}\right)\mu(\mathrm{d}x)\\ &\geq \mu\{x\,:\,|x|>2/u\}.\end{aligned}$$

■

We come back to the proof of tightness. Since ϕ is continuous at 0,

$$\frac{1}{u}\int_{-u}^{u}(1-\phi(t))\mathrm{d}t \to 0,$$

as $u \to 0$. Pick u_ε so that the integral is $< \varepsilon$. Since $\phi_n(t) \to \phi(t)$, by (BDD) for n large enough

$$\varepsilon \geq \frac{1}{u_\varepsilon}\int_{-u_\varepsilon}^{u_\varepsilon}(1-\phi_n(t))\mathrm{d}t \geq \mu_n\{x\,:\,|x|>2/u_\varepsilon\}.$$

Take $M_\varepsilon = 2/u_\varepsilon$ in the definition of tightness. That concludes the proof.

■

3 CLT: Simple form

We can now prove the CLT.

THM 6.29 *Let $(X_n)_n$ be IID with $\mathbb{E}[X_1] = \mu$ and $\mathrm{Var}[X_1] = \sigma^2 < +\infty$.Then if $S_n = \sum_{k\leq n} X_k$*

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow Z,$$

where $Z \sim \mathrm{N}(0,1)$.

Proof: Suffices to prove the result for $\mu = 0$. Note that

$$\phi_{X_1}(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2),$$

and by independence

$$\phi_{Z_n}(t) = \left(1 - \frac{t^2}{2n} + o(t^2)\right)^n \to e^{-t^2/2}.$$

The inversion formula and continuity theorem conclude the proof. (In fact, one must prove the above limit for complex numbers. This is done in [D].) ■

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