

# Appendix A

## Appendix

### A.1 Some definitions and useful facts

These lemmas will eventually be inserted in the main text in a suitable place.

#### A.1.1 Binomial coefficients

Recall the following bounds on binomial coefficients:

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{e^k n^k}{k^k},$$

$$\binom{2n}{n} = (1 + o(1)) \frac{4^n}{\sqrt{\pi n}},$$

and

$$\log \binom{n}{k} = (1 + o(1)) nH(k/n),$$

where  $H(p) := -p \log p - (1 - p) \log(1 - p)$ .

#### A.1.2 Conditional expectation: definition and properties

Recall the definition of the conditional expectation (see e.g. [Wil91, Section 9.2]).

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**Theorem A.1** (Conditional expectation). *Let  $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$  and  $\mathcal{G} \subseteq \mathcal{F}$  a sub  $\sigma$ -field. Then there exists a (a.s.) unique  $Y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$  (note the  $\mathcal{G}$ -measurability) s.t.*

$$\mathbb{E}[Y; G] = \mathbb{E}[X; G], \forall G \in \mathcal{G}.$$

*Such a  $Y$  is called a version of the conditional expectation of  $X$  given  $\mathcal{G}$  and is denoted by  $\mathbb{E}[X | \mathcal{G}]$ .*

In  $L^2$  conditional expectation reduces to an orthogonal projection (see e.g. [Wil91, Section 9.4]).

**Theorem A.2** (Conditional expectation:  $L^2$  case). *Let  $\langle X, Y \rangle := \mathbb{E}[XY]$ . Let  $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$  and  $\mathcal{G} \subseteq \mathcal{F}$  a sub  $\sigma$ -field. Then there exists a (a.s.) unique  $Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})$  s.t.*

$$\|X - Y\|_2 = \inf\{\|X - W\|_2 : W \in L^2(\Omega, \mathcal{G}, \mathbb{P})\},$$

*and, moreover,  $\langle Z, X - Y \rangle = 0, \forall Z \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ . Such  $Y$  is called an orthogonal projection of  $X$  on  $L^2(\Omega, \mathcal{G}, \mathbb{P})$ .*

In addition to linearity and the usual inequalities (e.g. Jensen's inequality, etc.) and convergence theorems (e.g. dominated convergence, etc.), we highlight the following three properties of the conditional expectation (see e.g. [Wil91, Section 9.7]).

**Lemma A.3** (Taking out what is known). *If  $Z \in \mathcal{G}$  is bounded then  $\mathbb{E}[ZX | \mathcal{G}] = Z \mathbb{E}[X | \mathcal{G}]$ . This is also true if  $X, Z \geq 0$  and  $\mathbb{E}[ZX] < +\infty$  or  $X \in L^p(\mathcal{F})$  and  $Z \in L^q(\mathcal{G})$  with  $p^{-1} + q^{-1} = 1$  and  $p > 1$ .*

**Lemma A.4** (Role of independence). *If  $X$  is independent of  $\mathcal{H}$  then  $\mathbb{E}[X | \mathcal{H}] = \mathbb{E}[X]$ . In fact, if  $\mathcal{H}$  is independent of  $\sigma(\sigma(X), \mathcal{G})$ , then  $\mathbb{E}[X | \sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X | \mathcal{G}]$ .*

**Lemma A.5** (Tower property (or law of total probability)). *We have  $\mathbb{E}[\mathbb{E}[X | \mathcal{G}]] = \mathbb{E}[X]$ . In fact, if  $\mathcal{H} \subseteq \mathcal{G}$  is a  $\sigma$ -field*

$$\mathbb{E}[\mathbb{E}[X | \mathcal{G}] | \mathcal{H}] = \mathbb{E}[X | \mathcal{H}].$$

*That is, the smallest  $\sigma$ -field wins.*

### A.1.3 A Taylor expansion

To be written. See [LL10, Lemmas 12.1.1, 12.1.4].

### A.1.4 Spectral representation of reversible matrices

Let  $P$  be the transition matrix of a finite, irreducible Markov chain on  $V$  reversible with respect to  $\pi$ . Define  $n := |V|$ . We let  $\ell^2(\pi)$  be the vector space of real-valued functions with inner product

$$\langle f, g \rangle_\pi := \sum_{x \in V} \pi(x) f(x) g(x).$$

**Lemma A.6** (Spectral representation: reversible matrices). *The space  $\ell^2(\pi)$  has an orthonormal basis of eigenfunctions  $\{f_j\}_{j=1}^n$  with real eigenvalues  $\{\lambda_j\}_{j=1}^n$  such that  $|\lambda_j| \leq 1$ , for all  $j$ . The eigenfunction  $f_1$  corresponding to the eigenvalue 1 can be taken to be the all-1 function. Furthermore, we have the following decomposition*

$$\frac{P^t(x, y)}{\pi(y)} = 1 + \sum_{j=2}^n f_j(x) f_j(y) \lambda_j^t.$$

*Proof.* To be written. See [LPW06, Lemma 12.2] ■

### A.1.5 A fact about trees

**Lemma A.7.** *A cycle-free undirected graph with  $n$  vertices and  $n - 1$  edges is a spanning tree.*

### A.1.6 A Poincaré inequality

The Dirichlet form is defined as  $\mathcal{E}(f, g) := \langle f, (I - P)g \rangle_\pi$ . Note that

$$\begin{aligned} & 2\langle f, (I - P)f \rangle_\pi \\ &= 2\langle f, f \rangle_\pi - 2\langle f, Pf \rangle_\pi \\ &= \sum_x \pi(x) f(x)^2 + \sum_y \pi(y) f(y)^2 - 2 \sum_x \pi(x) f(x) f(y) P(x, y) \\ &= \sum_{x, y} f(x)^2 \pi(x) P(x, y) + \sum_{x, y} f(y)^2 \pi(y) P(y, x) - 2 \sum_x \pi(x) f(x) f(y) P(x, y) \\ &= \sum_{x, y} f(x)^2 \pi(x) P(x, y) + \sum_{x, y} f(y)^2 \pi(x) P(x, y) - 2 \sum_x \pi(x) f(x) f(y) P(x, y) \\ &= \sum_{x, y} \pi(x) P(x, y) [f(x) - f(y)]^2 = 2\mathcal{E}(f) \end{aligned}$$

where

$$\mathcal{E}(f) := \frac{1}{2} \sum_{x, y} c(x, y) [f(x) - f(y)]^2,$$

is the Dirichlet energy encountered previously. We note further that if  $\sum_x \pi(x)f(x) = 0$  then

$$\langle f, f \rangle_\pi = \langle f - \langle \mathbf{1}, f \rangle_\pi, f - \langle \mathbf{1}, f \rangle_\pi \rangle_\pi = \text{Var}_\pi[f],$$

where the last expression denotes the variance under  $\pi$ . So the variational characterization of  $\lambda_2$  translates into

$$\text{Var}_\pi[f] \leq \gamma \mathcal{E}(f),$$

for all  $f$  such that  $\sum_x \pi(x)f(x) = 0$  (in fact for any  $f$  by considering  $f - \langle \mathbf{1}, f \rangle_\pi$  and noticing that both sides are unaffected by adding a constant), which is known as a *Poincaré inequality*.

**Lemma A.8** (Poincaré inequality).

$$\text{Var}_\pi[f] \leq \gamma \mathcal{E}(f), \quad \forall f,$$

with equality for  $f_2$ , the eigenfunction of  $P$  corresponding to the second largest eigenvalue  $\lambda_2$ .

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