Appendix A

Appendix

A.1 Some definitions and useful facts

These lemmas will eventually be inserted in the main text in a suitable place.

A.1.1 Binomial coefficients

Recall the following bounds on binomial coefficients:

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{e^k n^k}{k^k},$$

$$\binom{2n}{n} = (1 + o(1))\frac{4^n}{\sqrt{\pi n}},$$

and

$$\log \binom{n}{k} = (1 + o(1)) n H(k/n),$$

where $H(p) := -p \log p - (1-p) \log(1-p)$.

A.1.2 Conditional expectation: definition and properties

Recall the definition of the conditional expectation (see e.g. [Wil91, Section 9.2]).

Version: September 9, 2015
Modern Discrete Probability: An Essential Toolkit

Theorem A.1 (Conditional expectation)**.** *Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub σ-field. Then there exists a (a.s.) unique $Y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ (note the $\mathcal{G}$-measurability) s.t.*

$$\mathbb{E}[Y; G] = \mathbb{E}[X; G],\ \forall G \in \mathcal{G}.$$

Such a Y is called a version of the conditional expectation of X given $\mathcal{G}$ *and is denoted by* $\mathbb{E}[X \,|\, \mathcal{G}]$.

In L^2 conditional expectation reduces to an orthogonal projection (see e.g. [Wil91, Section 9.4]).

Theorem A.2 (Conditional expectation: L^2 case)**.** *Let $\langle X, Y \rangle := \mathbb{E}[XY]$. Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub σ-field. Then there exists a (a.s.) unique $Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ s.t.*

$$\|X - Y\|_2 = \inf\{\|X - W\|_2 \,:\, W \in L^2(\Omega, \mathcal{G}, \mathbb{P})\},$$

and, moreover, $\langle Z, X - Y \rangle = 0,\ \forall Z \in L^2(\Omega, \mathcal{G}, \mathbb{P})$. Such Y is called an orthogonal projection of X on $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

In addition to linearity and the usual inequalities (e.g. Jensen's inequality, etc.) and convergence theorems (e.g. dominated convergence, etc.), we highlight the following three properties of the conditional expectation (see e.g. [Wil91, Section 9.7]).

Lemma A.3 (Taking out what is known)**.** *If $Z \in \mathcal{G}$ is bounded then $\mathbb{E}[ZX \,|\, \mathcal{G}] = Z\,\mathbb{E}[X \,|\, \mathcal{G}]$. This is also true if $X, Z \geq 0$ and $\mathbb{E}[ZX] < +\infty$ or $X \in L^p(\mathcal{F})$ and $Z \in L^q(\mathcal{G})$ with $p^{-1} + q^{-1} = 1$ and $p > 1$.*

Lemma A.4 (Role of independence)**.** *If X is independent of $\mathcal{H}$ then $\mathbb{E}[X \,|\, \mathcal{H}] = \mathbb{E}[X]$. In fact, i If $\mathcal{H}$ is independent of $\sigma(\sigma(X), \mathcal{G})$, then $\mathbb{E}[X \,|\, \sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X \,|\, \mathcal{G}]$.*

Lemma A.5 (Tower property (or law of total probability))**.** *We have $\mathbb{E}[\mathbb{E}[X \,|\, \mathcal{G}]] = \mathbb{E}[X]$. In fact, if $\mathcal{H} \subseteq \mathcal{G}$ is a σ-field*

$$\mathbb{E}[\mathbb{E}[X \,|\, \mathcal{G}] \,|\, \mathcal{H}] = \mathbb{E}[X \,|\, \mathcal{H}].$$

That is, the smallest σ-field wins.

A.1.3 A Taylor expansion

To be written. See [LL10, Lemmas 12.1.1, 12.1.4].

A.1.4 Spectral representation of reversible matrices

Let P be the transition matrix of a finite, irreducible Markov chain on V reversible with respect to π. Define $n := |V|$. We let $\ell^2(\pi)$ be the vector space of real-valued functions with inner product

$$\langle f, g \rangle_\pi := \sum_{x \in V} \pi(x) f(x) g(x).$$

Lemma A.6 (Spectral representation: reversible matrices)**.** *The space $\ell^2(\pi)$ has an orthonormal basis of eigenfunctions $\{f_j\}_{j=1}^n$ with real eigenvalues $\{\lambda_j\}_{j=1}^n$ such that $|\lambda_j| \leq 1$, for all j. The eigenfunction f_1 corresponding to the eigenvalue 1 can be taken to be the all-1 function. Furthermore, we have the following decomposition*

$$\frac{P^t(x,y)}{\pi(y)} = 1 + \sum_{j=2}^n f_j(x) f_j(y) \lambda_j^t.$$

Proof. To be written. See [LPW06, Lemma 12.2] ■

A.1.5 A fact about trees

Lemma A.7. *A cycle-free undirected graph with n vertices and $n-1$ edges is a spanning tree.*

A.1.6 A Poincaré inequality

The *Dirichlet form* is defined as $\mathcal{E}(f,g) := \langle f, (I-P)g \rangle_\pi$. Note that

$$\begin{aligned}
&2\langle f, (I-P)f \rangle_\pi \\
&\quad = 2\langle f, f \rangle_\pi - 2\langle f, Pf \rangle_\pi \\
&\quad = \sum_x \pi(x) f(x)^2 + \sum_y \pi(y) f(y)^2 - 2\sum_x \pi(x) f(x) f(y) P(x,y) \\
&\quad = \sum_{x,y} f(x)^2 \pi(x) P(x,y) + \sum_{x,y} f(y)^2 \pi(y) P(y,x) - 2\sum_x \pi(x) f(x) f(y) P(x,y) \\
&\quad = \sum_{x,y} f(x)^2 \pi(x) P(x,y) + \sum_{x,y} f(y)^2 \pi(x) P(x,y) - 2\sum_x \pi(x) f(x) f(y) P(x,y) \\
&\quad = \sum_{x,y} \pi(x) P(x,y) [f(x) - f(y)]^2 = 2\mathcal{E}(f)
\end{aligned}$$

where

$$\mathcal{E}(f) := \frac{1}{2} \sum_{x,y} c(x,y) [f(x) - f(y)]^2,$$

is the Dirichlet energy encountered previously. We note further that if $\sum_x \pi(x) f(x) = 0$ then

$$\langle f, f \rangle_\pi = \langle f - \langle \mathbf{1}, f \rangle_\pi, f - \langle \mathbf{1}, f \rangle_\pi \rangle_\pi = \mathrm{Var}_\pi[f],$$

where the last expression denotes the variance under π. So the variational characterization of λ_2 translates into

$$\mathrm{Var}_\pi[f] \leq \gamma \mathcal{E}(f),$$

for all f such that $\sum_x \pi(x) f(x) = 0$ (in fact for any f by considering $f - \langle \mathbf{1}, f \rangle_\pi$ and noticing that both sides are unaffected by adding a constant), which is known as a *Poincaré inequality*.

Lemma A.8 (Poincaré inequality)**.**

$$\mathrm{Var}_\pi[f] \leq \gamma \mathcal{E}(f), \qquad \forall f,$$

with equality for f_2, the eigenfunction of P corresponding to the second largest eigenvalue λ_2.

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Index